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Título de la tesis:
Bootstrapping Unobserved Component Models

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Directora: Esther Ruiz

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To my family and my wife Fernanda

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Resumen

La utilidad de los modelos de componentes inobservados ha sido probada en innumerables oportunidades mediante trabajos empíricos, no sólo para explicar la evolución dinámica de las series, sino la de los componentes inobservados, los cuales, muchas veces, tienen su interés en sí mismos; por ejemplo, ver Orphanides and van Norden (2002), Smets (2002), Gerlach and Yiu (2004) y Doménech and Gómez (2006) para la estimación del "output gap" en varias economías, Harvey (2008) para la modelización de la curva de Phillips en los EEUU, Alonso et al. (2008) para explicar la evolución de los precios españoles de la electricidad, Stock and Watson (2007) para un modelo de componentes cíclicos donde la inflación posee componentes de volatilidad estocástica.

En un contexto de modelos lineales de componentes inobservados con errores condicionalmente Gaussianos, si los parámetros son asumido conocidos, el filtro de Kalman proporciona estimaciones de tales componentes con error cuadrático medio mínimo (ECMM) conjuntamente con sus correspondientes errores cuadráticos medios (ECM) condicionados a la información disponible; ver Harvey (1989). Sin embargo, en la práctica, los parámetros son desconocidos y tienen que ser estimados por algún procedimiento consistente. La estimación de los parámetros introduce una nueva fuente de incertidumbre en la estimación de los componentes inobservados. Consecuentemente, cuando los ECMs de las predicciones y de los componentes inobservados son estimados usando el filtro de Kalman ejecutado con parámetros estimados, ellos subestiman la

verdadera incertidumbre de tal estimación.

Varios procedimientos han sido propuestos para incorporar la incertidumbre de la estimación de los parámetros, ellos se pueden agrupar en tres conjuntos. El primero, en el cual los procedimientos están basados en técnicas Bayesianas. Sin embargo, estos procedimientos pueden ser computacionalmente muy intensivos y requieren mucho tiempo de cálculo. El segundo grupo contiene aquellos procedimientos que se basan en incorporar la incertidumbre de la estimación de los parámetros utilizando la distribución asintótica del estimador. No obstante, en muestra pequeñas, esta distribución puede ser una muy pobre aproximación de la verdadera distribución del estimador. Finalmente, encontramos los procedimientos que utilizan las técnicas bootstrap para aproximar la distribución del estimador en muestra pequeñas. Analizando los procedimientos bootstrap para la incorporación de la incertidumbre de estimación hemos encontrado dos problemas importantes en ellos. Por un lado, cuando el objetivo es el cómputo del ECM de la estimación de los componentes inobservados, los procedimientos bootstrap aproximan los ECMs incondicionales, en el sentido de que no están estimados condicionados a la información disponible. Cabe mencionar, que el filtro de Kalman provee estimaciones condicionales tanto de los componentes como de sus ECMs. Por otro lado, cuando el objetivo es la construcción de intervalos de predicción de las series observadas, los procedimientos bootstrap están basados en la representación "backward" del modelo, y por lo tanto, aquellos modelos para los que esta representación no existe, estos no se pueden aplicar, además, son computacionalmente complicados.

En esta tesis, proponemos el uso de técnicas bootstrap para incorporar la incertidumbre de la estimación de los parámetros en modelos de componentes inobservados expresado en un contexto de modelos de espacio de los estados. A lo largo de los capítulos usamos simulaciones Monte Carlo y datos reales para mostrar los resultados de los procedimientos propuestos.

En el Capítulo 2 proponemos dos procedimientos bootstrap para la estimación del ECM de los componentes inobservados que provee el filtro de Kalman, incorporando la incertidumbre de la estimación de los parámetros, y además, son condicionales a la información disponible. Por otro lado, mostramos que nuestros procedimientos son más simples que los alternativos propuestos en la literatura. Finalmente, mostramos en una aplicación empírica que cuando se incorpora la incertidumbre de la estimación de los parámetros, las conclusiones de política económica pueden cambiar respecto a la situación en la que no se considera tal incertidumbre.

En el Capítulo 3 proponemos un procedimiento bootstrap para la aproximación de las densidades de las observaciones futuras de la serie observada, y por lo tanto, la construcción de intervalos de predicción. Nuestro procedimiento no está basado en la representación “backward” del modelo, por lo que se puede extender a aquellos modelos para los cuales esta representación no existe. Además, debido a que no necesita esta representación, es computacionalmente más simple, y por lo tanto, requiere menor tiempo de cálculo. Finalmente, mediante simulaciones Monte Carlo, mostramos que nuestro procedimiento tiene mejores propiedades en muestras pequeñas en un modelo de nivel local.

Chapter 1

Introduction

1.1 Motivation

Unobserved component (UC) models have proven to be very useful for describing the dynamic evolution of financial and economic time series; see, for example, Orphanides and van Norden (2002), Smets (2002), Gerlach and Yiu (2004) and Doménech and Gómez (2006) for estimating the output gap in several economies, Harvey (2008) for modelling the Phillips curve in the US economy, Alonso et al. (2008) for Spanish electricity prices or Pedregal and Young (2006) for electricity load demand with unobserved modulated periodic components, Stock and Watson (2007) for a trend-cycle model with stochastic volatility fitted to US inflation and Malley and Molana (2008) for a model for unemployment. On the other hand, when modelling financial series, the volatility can also be modelled as an unobserved component; such models are referred as stochastic volatility models; see Taylor (1986), Harvey et al. (1994), Ghysels et al. (1996), Barndorff-Nielsen et al. (2002) and Carnero et al. (2004) for some useful references describing these models.

One of the main attractiveness of UC models is that they allow the estimation of the underlying components which are often of interest in themselves. In the context of linear and Gaussian UC models, if the parameters are assumed to be known, the Kalman filter provides Minimum Mean Square Error (MMSE) estimates of the underlying components together with their corresponding prediction mean square errors (PMSE) conditional on the available information set; see Harvey (1989). In this sense, the filter provides one-step-ahead predictions of the underlying components at time t conditional on the information up to time $t-1$. It also provides updated estimates of the underlying components at time t based on the information available up to time t. Finally, the Kalman filter provides the information necessary for obtaining estimates of the underlying components conditional on the whole sample using an algorithm known as smoothing; see Durbin and Koopman (2001). Moreover, the filter also gives the framework for making k-step-ahead predictions and construct prediction intervals for future values of the observed series and of the underlying unobserved components given the available data set.

As we mentioned, the Kalman filter is run assuming known parameters. However, in practice, the model parameters are unknown and they must be substituted by consistent estimates. Consequently, this introduces a new source of uncertainty associated with parameter estimation that should be introduced in the PMSE of both the underlying components and future observations. Consequently, when both PMSEs are computed by the standard expressions given by the Kalman filter with the true parameter substituted by their estimates, they underestimate the true uncertainty.

There are several alternatives proposed in the literature for incorporating the parameter estimation uncertainty into the PMSEs given by the Kalman filter. Consider first, the procedures proposed to incorporate the estimation uncertainty into the prediction intervals of future values of the observed variables. Several authors propose Bayesian procedures which provide the posterior distribution of the parameters conditional on the available information which is used for estimating the PMSE; see for example Liang and Kelemen (2007) and Pedroza (2006). However, these procedures can be computational very intensive and very hard of dealing with when the structure behind the model is quite complicated; see for example Datta et al. (1999). Alternatively, the parameter uncertainty can be incorporated into the PMSEs by using the asymptotic distribution of the parameters estimator to approximate its finite empirical distribution. The main problem with these procedures is that the asymptotic distribution may be a very poor approximation in finite samples of the true finite distribution of the parameter estimator. Finally, an alternative that has proved to have very good performance in practical applications is to use bootstrap procedures. These procedures, introduced by Efron (1979), are based on resampling data coming from a variable that is independent and identically distributed (IID). Although, in time series data this aspect is clearly violated, there are several bootstrap procedures that deal with dependent observations. One solution is resampling the residuals from of the estimated model associated with the data generating process (DGP). Therefore, by resampling those residuals and using the chosen model with the estimated parameters, one can obtain bootstrap replicates that mimic the dynamic properties of the original observations; see for instance, Li and Maddala (1996) and Hardle et al. (2003) for very interesting reviews of bootstrap procedures in time series. Although, bootstrap procedures seem to be an interesting alternative for incorporating the parameter uncertainty into the PMSEs provided by the Kalman filter, the current procedures in the literature are based on the backward representation of the model. In particular, Wall and Stoffer (2002) propose a bootstrap procedure for incorporating the parameter uncertainty into the prediction intervals of future values of the series of interest by fixing the last observation of each bootstrap replicate. Consequently, the procedure requires the backward representation of the model and its application is limited to those models for which that representation exists. Moreover, the procedure proposed by Wall and Stoffer (2002) does not approximate the prediction densities of the future values of the series, but that of the prediction errors which complicates even further the procedure. In this thesis, following the ideas of Pascual et al. (2004), we propose a new bootstrap procedure to incorporate the parameter uncertainty into the prediction intervals of future observations that does not rely on the backward representation of the model. Consequently, the new bootstrap procedure is much simpler from a computational point of view without loosing the good finite sample performance of bootstrap procedures. Furthermore, it can be implemented in models in which the backward representation does not exists.

Bayesian, asymptotic and bootstrap procedures have also been proposed to incorporate the parameter uncertainty into the PMSE associated with the estimates of the underlying components of the model; see for example, Ansley and Kohn (1986), Hamilton (1986) and Quenneville and Singh (2000) for the asymptotic and Bayesian approximations. However, the asymptotic and Bayesian procedures have the same limitations commented for the prediction intervals. Recently, Pfeffermann and Tiller (2005) propose several bootstrap procedures for incorporating the parameter uncertainty into the PMSEs of the estimated unobserved states. The bootstrap PMSEs proposed by Pfeffermann and Tiller (2005) are based on obtaining the unconditional PMSEs of the estimates of the underlying states. However, one should note that the Kalman filter is designed to generate PMSE conditional on the available information set. Although this distinction is irrelevant in state space models with time-invariant system matrices, it could be important when the system matrices in state space model are observation dependent. Consequently, by taking into account this distinction, in this thesis, we propose new bootstrap procedures that simplify the bootstrap procedures proposed by

Pfeffermann and Tiller (2005) improving, at the same time, their finite sample performance.

It is also important to mention that bootstrap procedures not only allow to incorporate the parameter uncertainty into PMSEs but also, they do not require, in general, the assumption of conditional Gaussianity for their implementation. Therefore, they allow to obtain prediction intervals of future observations and PMSEs of the underlying components in the context of non-Gaussian models.

The rest of the chapter is organized as follows. Section 2 describes briefly the properties of state space models and the Kalman filter needed in the rest of the thesis. We also describe how prediction intervals of future observations can be constructed using the information that the filter provides. Finally, we also briefly describe the asymptotic procedure of Hamilton (1986) for incorporating the parameter uncertainty into the PMSEs of the estimated unobserved components. Section 3 describes the bootstrap procedures previously proposed in the literature for incorporating the parameter uncertainty into PMSEs of estimated underlying states and prediction intervals of future observations. Finally, Section 4 summarizes the main objectives of this thesis and its organization.

1.2 State space models and the Kalman filter

In this section we describe state space models and the Kalman filter. First, we describe the filter run with known parameters. Then, we present the problem of using the estimated parameters instead of the true ones for estimating the PMSE of the estimated unobserved components. Finally, we describe how prediction intervals of the future values of the series are constructed using the filter.

1.2.1 The filter with known and estimated parameters

Unobserved component models can be casted into the following state space framework

$$Y_t = Z_t\alpha_t + d_t + R_{1t}\varepsilon_t, \tag{1.1a}$$

$$\alpha_t = T_t\alpha_{t-1} + c_t + R_{2t}\eta_t, \quad t = 1, \ldots, T, \tag{1.1b}$$

where Y_t is a $N \times 1$ vector time series observed at time t, α_t is the $m \times 1$ vector of unobservable state variables, ε_t is a $k \times 1$ vector of independent white noises with zero mean and covariance matrix H_t and η_t is a $g \times 1$ vector of serially uncorrelated disturbances with zero mean and covariance matrix Q_t. The disturbances ε_t and η_t are uncorrelated with each other in all time periods. Finally, the initial state vector, α_1, has mean $a_{1|0}$ and covariance matrix $P_{1|0}$. All the system matrices, Z_t, d_t, R_{1t}, T_t, c_t, R_{2t}, H_t and Q_t, are assumed to be known one-step-ahead. The model in (1.1) is time-invariant when, with the exception of d_t and c_t, all the system matrices are time-invariant.

The Kalman filter provides one-step-ahead estimates at time t of the underlying states, α_t, and their corresponding PMSE, which are denoted by $a_{t|t-1}$ and $P_{t|t-1}$ respectively, given the information available at time $t-1$, i.e. $\{Y_1, ..., Y_{t-1}\}$. If the errors are further assumed to have a conditional joint Normal distribution, using well known results of this distribution, $a_{t|t-1}$ and $P_{t|t-1}$ are the conditional mean and conditional PMSE respectively. In particular, it is possible to see that, in this case,

$$\left.\begin{array}{c} \alpha_t \\ Y_t \end{array}\right| Y_1, \ldots, Y_{t-1} \sim \mathcal{N}\left[\left(\begin{array}{c} a_{t|t-1} \\ Z_t a_{t|t-1} + d_t \end{array}\right), \left(\begin{array}{cc} P_{t|t-1} & P_{t|t-1}Z_t' \\ Z_t P_{t|t-1} & F_t \end{array}\right)\right], \tag{1.2}$$

where $a_{t|t-1}$ and $P_{t|t-1}$ are given by the following prediction equations

$$a_{t|t-1} = T_t a_{t-1} + c_t \tag{1.3a}$$

$$P_{t|t-1} = T_t P_{t-1} T_t' + R_{2t} Q_t R_{2t}', \tag{1.3b}$$

and a_{t-1} and P_{t-1} are given by the following updating equations

$$a_{t-1} = a_{t-1|t-2} + P_{t-1|t-2}Z'_{t-1}F_{t-1}^{-1}V_{t-1}, \tag{1.3c}$$

$$P_{t-1} = P_{t-1|t-2} - P_{t-1|t-2}Z'_{t-1}F_{t-1}^{-1}P_{t-1|t-2}Z'_{t-1} + R_{2t-1}Q_tR'_{2t-1}, \tag{1.3d}$$

with $V_t = Y_t - Z_t a_{t|t-1} - d_t$ the one-step-ahead vector of innovations and $F_t = Z_t P_{t|t-1} Z'_t + R_{1t}H_tR'_{1t}$ its covariance matrix; see, Harvey (1989) for details. When running the Kalman filter equations given in (1.3), it is assumed that all the parameters involved in the system matrices and the initial conditions $a_{1|0}$ and $P_{1|0}$ are known. It is important to observe that in linear models in which the system matrices are independent of the observations, the PMSE, $P_{t|t-1}$, is also independent of the observations. Therefore, in this case, $P_{t|t-1}$ is also the unconditional error covariance matrix associated with the conditional mean estimator of the underlying state. Also note that, when the system of matrices are time-invariant, the Kalman filter converges to a steady state with covariance matrices $P_{t|t-1} = \overline{P}$ and $P_t = a\overline{P}$, where a is a constant, and $F_t = \overline{F}$; see Anderson and Moore (1979) and Harvey (1989). Finally, note that, when the model in (1.1) is not conditionally Gaussian, the Kalman filter in (1.3) is still optimum in the sense that it provides the Minimum Mean Square Linear Estimator (MMSLE) of the underlying components; see Harvey (1989).

Finally, it is also useful for one of the bootstrap procedures described later in this thesis to express the state space model in (1.1) in what is known as the Innovation Form (IF) which depends on a unique disturbance vector instead of two. The IF is given by equation (1.3a) together with

$$Y_t = Z_t a_{t|t-1} + d_t + V_t. \tag{1.4}$$

Note that the unique disturbance vector in the IF is the one-step-ahead innovations, V_t.

Up to now, we have assumed that the parameters of the model are known when the Kalman filter is run. However, in practice, some of these parameters are unknown and have to be substituted by consistent estimates. In this thesis, we consider the Quasi-Maximum Likelihood (QML) estimator of the parameters based on maximizing the Gaussian likelihood; see, for example, Harvey (1989) and Durbin and Koopman (2001) for details. Denote by $\widehat{Z}_t, \widehat{d}_t, \widehat{H}_t, \widehat{R}_{1t}, \widehat{T}_t, \widehat{c}_t, \widehat{R}_{2t}$ and $\widehat{Q}_t$ the system of matrices where the unknown parameters have been substituted by their QML estimates. Furthermore, the initial conditions for the filter are also unknown. The usual practice is to assume that they are given by the unconditional distribution of the unobserved state in case of stationary states or by a diffuse prior distribution when they are non-stationary; see Harvey (1989). Then, equations (1.3a)-(1.3d) of the Kalman filter can be run with the system matrices substituted by their respective estimates providing $\widehat{a}_{t|t-1}$ and $\widehat{P}_{t|t-1}$ respectively, and the corresponding updated estimates $\widehat{a}_t$ and $\widehat{P}_t$. Note that, $\widehat{a}_{t|t-1}$ is an estimate of the conditional mean of the underlying component, $a_{t|t-1}$. However, $\widehat{P}_{t|t-1}$ is not the PMSE of $\widehat{a}_{t|t-1}$ because it does not take into account the parameter estimation uncertainty.

1.2.2 PMSE of the unobserved states with estimated parameters

As we mentioned, when the Kalman filter is run with the system matrices substituted by their respective estimates, it provides $\widehat{a}_{t|t-1}$ and $\widehat{P}_{t|t-1}$ where, $\widehat{a}_{t|t-1}$, is an estimate of the conditional mean of the state, $a_{t|t-1}$, but, $\widehat{P}_{t|t-1}$ is no longer the PMSE of $\widehat{a}_{t|t-1}$, which is given by

$$
\begin{aligned}
PMSE_{t|t-1} &= \underset{t-1}{E}\left[\left(\widehat{a}_{t|t-1}-\alpha_t\right)\left(\widehat{a}_{t|t-1}-\alpha_t\right)'\right] \\
&= \underset{t-1}{E}\left[\left(\widehat{a}_{t|t-1}-a_{t|t-1}\right)\left(\widehat{a}_{t|t-1}-a_{t|t-1}\right)'\right]+ \\
&\quad \underset{t-1}{E}\left[\left(a_{t|t-1}-\alpha_t\right)\left(a_{t|t-1}-\alpha_t\right)'\right] \\
&= \underset{t-1}{E}\left[\left(\widehat{a}_{t|t-1}-a_{t|t-1}\right)\left(\widehat{a}_{t|t-1}-a_{t|t-1}\right)'\right]+P_{t|t-1} \qquad (1.5)
\end{aligned}
$$

where the $t-1$ under the expectation means that it is taken conditional on $\{Y_1,\ldots,Y_{t-1}\}$. Note that the cross-product $\underset{t-1}{E}\left[\left(\widehat{a}_{t|t-1}-a_{t|t-1}\right)\left(a_{t|t-1}-\alpha_t\right)'\right]$ is zero under the assumption of conditional Normality. The second term in (1.5) is denoted by Hamilton (1986) as filter uncertainty. It represents how far would the state be from its estimate when the parameters are known. This uncertainty is due to the uncertainty in separating signal and noise and it is inherent to the Kalman filter. On the other hand, the first term in (1.5), denoted as parameter uncertainty, represents the discrepancy between the estimates of the unobserved states obtained with known and unknown parameters. $\widehat{P}_{t|t-1}$ does not take into account the parameter uncertainty involved in the first term of (1.5). Therefore, $\widehat{P}_{t|t-1}$ will underestimate in general the true conditional PMSE of $\widehat{a}_{t|t-1}$.

As we said in the introduction, there have been several proposals in the literature to compute the PMSE of the estimator of the unobserved components that take into account the parameter uncertainty. Next, we describe one of the most popular of these alternatives proposed by Hamilton (1986) which is based on the asymptotic distribution of the parameter estimator. Hamilton (1986) proposes to estimate the PMSE of $\widehat{a}_{t|t-1}$ by considering the decomposition in (1.5) and the following relationship

$$
\begin{aligned}
PMSE_{t|t-1} &= E_\theta\left\{\underset{t-1}{E}\left[\left(\widehat{a}_{t|t-1}-a_{t|t-1}\right)\left(\widehat{a}_{t|t-1}-a_{t|t-1}\right)'|\theta\right]\right\}+ \\
&\quad E_\theta\left\{\underset{t-1}{E}\left[\left(a_{t|t-1}-\alpha_t\right)\left(a_{t|t-1}-\alpha_t\right)'|\theta\right]\right\}, \qquad (1.6)
\end{aligned}
$$

where θ is the vector of model parameters.

Once the parameters are estimated, a large number, M, of realizations of $\widehat{\theta}^{(i)}$ are generated from the asymptotic distribution of the estimator. Then, the Kalman filter is run as in 1.3 using each of the realizations $\widehat{\theta}^{(i)}$ and the original observations, $\{Y_1, \ldots, Y_T\}$, obtaining a series of estimates the state and their corresponding PMSE, denoted by $\widehat{a}^{(i)}_{t|t-1}$ and $\widehat{P}^{(i)}_{t|t-1}$, respectively. In this way, an analogue of the expectations within squared brackets in (1.6) can be obtained by $\left(\widehat{a}^{(i)}_{t|t-1} - \widehat{a}_{t|t-1}\right)\left(\widehat{a}^{(i)}_{t|t-1} - \widehat{a}_{t|t-1}\right)'$ and $\widehat{P}^{(i)}_{t|t-1}$ respectively. Then, the sample averages for all possible values of the parameters are obtained to estimate the expectation over all values of θ. Finally, the estimate of PMSE in (1.6) is given by

$$\widehat{PMSE}^{Asy}_{t|t-1} = \frac{1}{M}\sum_{i=1}^{M}\widehat{P}^{(i)}_{t|t-1} + \frac{1}{M}\sum_{i=1}^{M}\left(\widehat{a}^{(i)}_{t|t-1} - \widehat{a}_{t|t-1}\right)\left(\widehat{a}^{(i)}_{t|t-1} - \widehat{a}_{t|t-1}\right)'. \tag{1.7}$$

1.2.3 Prediction intervals of future observations

Consider the state space model in (1.1) with time-invariant system of matrices, then, as mentioned before, the Kalman filter converges to a steady state with covariance matrices $P_{t|t-1} = \overline{P}$ and $P_t = a\overline{P}$, where a is a constant, and $F_t = \overline{F}$. After the last observation is available, the Kalman filter can still be run without the updating equations, in (1.3c) and (1.3d). In this case, the k-step ahead predictions of the underlying unobserved components are given by

$$a_{T+k|T} = T^k a_T + \sum_{j=0}^{k-1} T^j c, \tag{1.8a}$$

and their associated PMSE matrices are given by

$$P_{T+k|T} = \left(T^k\right) P_T \left(T^k\right)' + \sum_{j=0}^{k-1}\left[\left(T^j\right) R_2 Q R_2' \left(T^j\right)'\right]. \tag{1.8b}$$

The k-step ahead prediction of Y_{T+k} is given by

$$\widetilde{Y}_{T+k|T} = Z a_{T+k|T} + d, \tag{1.9a}$$

with prediction PMSE given by

$$PMSE(\widetilde{Y}_{T+k|T}) = ZP_{T+k|T}Z' + R_1HR_1'. \tag{1.9b}$$

Consequently, assuming that future prediction errors are Normally distributed, prediction intervals for Y_{T+k} are given by

$$\left[\widetilde{Y}^{(i)}_{T+k|T} - z_{1-\alpha/2}\sigma^{(i)}_{T+k|T},\ \widetilde{Y}(i)_{T+k|T} + z_{1-\alpha/2}\sigma^{(i)}_{T+k|T}\right],\ i = 1, \ldots, N, \tag{1.10}$$

where $\sigma^{(i)}_{T+k|T}$ is the ith element of the main diagonal of $PMSE(\widetilde{Y}_{T+k|T})$ in (1.9b) and $z_{1-\alpha/2}$ is the $\left(1-\frac{\alpha}{2}\right)$-percentile of the Standard Normal distribution; see, for example, Durbin and Koopman (2001).

When the parameters of the model are estimated, the prediction and their PMSE are obtained by (1.9) with all system matrices substituted by their estimates and with $a_{T+k|T}$ and $P_{T+k|T}$ substituted by $\widehat{a}_{T+k|T}$ and $\widehat{P}_{T+k|T}$ respectively, where the latter are given by (1.8a) with the system matrices substituted by their estimates and a_T and p_T substituted by $\widehat{a}_T$ and $\widehat{P}_T$ respectively. Consequently, the k-step ahead prediction of Y_{T+k} is given by

$$\widehat{Y}_{T+k|T} = \widehat{Z}_{T+k}\widehat{a}_{T+k|T} + \widehat{d}_{T+k} \tag{1.11a}$$

with estimated PMSE given by

$$\widehat{PMSE}\left(\widehat{Y}_{T+k|T}\right) = \widehat{Z}\widehat{P}_{T+k|T}\widehat{Z}' + \widehat{R}_1\widehat{H}\widehat{R}_1'. \tag{1.11b}$$

Then, in practice, the prediction intervals for future values of Y_t are given by

$$\left[\widehat{Y}^{(i)}_{T+k|T} - z_{1-\alpha/2}\widehat{\sigma}^{(i)}_{T+k|T},\ \widehat{Y}^{(i)}_{T+k|T} + z_{1-\alpha/2}\widehat{\sigma}^{(i)}_{T+k|T}\right],\ i = 1, \ldots, N. \tag{1.12}$$

where $\sigma^{(i)}_{T+k|T}$ is the ith element of the main diagonal of $\widehat{PMSE}(\widehat{Y}_{T+k|T})$ in (1.11b). We denote the interval in (1.12) as standard (ST). Note that $\widehat{P}_{T+k|T}$ does not incorporate the parameter uncertainty and, consequently, the ST prediction intervals in (1.12) are expected to have coverages under the nominal. Furthermore, they are constructed under the assumption of conditional Normality. Therefore, they can be inadequate when this assumption is not satisfied.

1.3 Bootstrap procedures for state space models

In this section we describe the bootstrap procedures available in the literature proposed to incorporate the parameter estimation uncertainty into the PMSEs of the estimated unobserved states and prediction intervals of future values of the series of interest.

1.3.1 Bootstrap procedures for estimating PMSE

As we mentioned, when the Kalman filter is run with estimated parameters the PMSE associated with the estimates of the underlying component, $\widehat{P}_{t|t-1}$, underestimate the true uncertainty of $\widehat{a}_{t|t-1}$. Hamilton (1986) propose to use the asymptotic distribution of the estimator to incorporate the parameter uncertainty into the PMSEs. However, when the sample size is small or even moderate, this approximation can be poor; see for example Quenneville and Singh (2000). To overcome this problem, Pfeffermann and Tiller (2005) propose to use bootstrap procedures. Although, the Kalman filter is designed to generate PMSE conditional on the available information set, their are based on obtaining the unconditional PMSE of estimates of the underlying states. They propose parametric and non-parametric bootstrap procedures. Next, we only describe the parametric bootstrap because it has the best performance according to our simulation results. They consider the decomposition of the PMSE in (1.5) but with the expectations taken over all possible realizations of $\{Y_1, \ldots, Y_T\}$ and $\{\alpha_1, \ldots, \alpha_T\}$ instead of expectations conditional on the available data set. The parametric bootstrap analogue of (1.5) proposed by Pfeffermann and Tiller (2005) is obtained as follows:

Step 1: Given a realization of $\{Y_1, \ldots, Y_T\}$, estimate the parameters, $\widehat{\theta}$, and implement the Kalman filter to obtain the estimates of the underlying state, $\widehat{a}_{t|t-1}(\widehat{\theta})$ and the corresponding PMSE, $\widehat{P}_{t|t-1}(\widehat{\theta})$.[1]

Step 2: Obtain a bootstrap replicate of the series $\{Y_1^*, \ldots, Y_T^*\}$, and of the underlying state $\{\alpha_1^*, \ldots, \alpha_T^*\}$, by extracting realizations, ε_t^* and η_t^*, $t = 1, \ldots, T$, from the joint Gaussian distribution of ε_t and η_t, using them and the estimated parameters, $\widehat{\theta}$, substituted in model (1.1). Then, estimate the bootstrap parameters, $\widehat{\theta}^*$.

Step 3: Implement again the Kalman filter with the bootstrap estimates, $\widehat{\theta}^*$, and the bootstrap replicates $\{Y_1^*, \ldots, Y_T^*\}$, to obtain bootstrap estimates of the state, $\widehat{a}^*_{t|t-1}(\widehat{\theta}^*)$, and their corresponding PMSE, $\widehat{P}^*_{t|t-1}(\widehat{\theta}^*)$.

Step 4: Using the bootstrap series $\{Y_1^*, \ldots, Y_T^*\}$ and the parameters estimated in step 1, $\widehat{\theta}$, run the Kalman filter to obtain the estimates of the state denoted by $\widehat{a}^*_{t|t-1}(\widehat{\theta})$.

[1]We add explicitly the dependence of the estimates of the unobserved states and their corresponding PMSE on the estimated parameters to clarify the procedure.

Repeat B times steps 2 to 4. Finally, the bootstrap analogue of the PMSE of $\widehat{a}_{t|t-1}$ in (1.5) is estimated by[2]

$$\widehat{PMSE}_t^{PT} = \frac{1}{B}\sum_{j=1}^{B}\left(\widehat{a}_{t|t-1}^{*(j)}(\widehat{\theta}^*) - \widehat{a}_{t|t-1}^{*}(\widehat{\theta})\right)\left(\widehat{a}_{t|t-1}^{*(j)}(\widehat{\theta}^*) - \widehat{a}_{t|t-1}^{*}(\widehat{\theta})\right)' + 2\widehat{P}_{t|t-1} - \frac{1}{B}\sum_{j=1}^{B}\widehat{P}_{t|t-1}^{*(j)}(\widehat{\theta}^*). \tag{1.13}$$

As we noted above, the PMSE in (1.13) is computed by taking expectations over all bootstrap realizations of the original series. However, the Kalman filter is designed to obtain conditional estimates of the underlying state and their corresponding PMSE. Therefore, it could be possible to simplify computationally the bootstrap procedure and, simultaneously improve its performance by computing the PMSEs conditional on the available data set.

1.3.2 Bootstrap prediction intervals of future observations

In the context univariate of ARIMA models several authors propose to use bootstrap procedures to construct prediction intervals that incorporate the parameter uncertainty. The seminal paper in this area is Thombs and Schucany (1990) who propose a bootstrap procedure to obtain prediction intervals for $AR(p)$ models based on estimating directly the distribution of the conditional predictions. They argue that, because predictions are conditional on the available data set, all bootstrap replicates generated to obtain bootstrap replicates of the estimated parameters, should have the same last p values. Consequently, the procedure of Thombs and Schucany (1990) requires the use of the

[2]Pfeffermann and Tiller (2005) also propose a non-parametric bootstrap for estimating the PMSE which is based on obtaining the bootstrap replicates of $\{Y_1^*, \ldots, Y_T^*\}$ by using the IF of the model in (1.3a) and (1.4) and random extractions, $\{V_1^*, \ldots, V_T^*\}$, from the empirical distribution of the standardized innovations, $\widehat{V}_t\widehat{F}_t^{-1/2}$; see, Stoffer and Wall (1991) and Rodriguez and Ruiz (2009) for its practical implementation. This non-parametric bootstrap does not assume any particular distribution of the errors. In our comparisons, we do not consider this non-parametric bootstrap because the results are always worse than those of the parametric bootstrap.

backward representation of the model. The need of this representation complicates computationally the procedure and limits its implementation to models with it. On the other hand, Pascual et al. (2004) show that when trying to incorporate parameter uncertainty in prediction intervals, there is not need of fixing the last p observations of each bootstrap replicate. They only fix the last p observations to obtain bootstrap replicates of future values of the series but the estimated parameters are bootstrapped without fixing any observation in the sample. Consequently, the backward representation is unnecessary, which simplifies the construction of bootstrap prediction intervals and allows to extend the procedure to models without such representation. Unlike ARIMA models, models with unobserved components may have several disturbances. Therefore, the bootstrap procedures proposed by Thombs and Schucany (1990) and Pascual et al. (2004) cannot be directly applied to them. To overcome this problem, Stoffer and Wall (1991) propose to use the IF based on a unique set of error terms, in order to use bootstrap procedures for obtaining prediction intervals of future observations in state space models. However, as in Thombs and Schucany (1990), the bootstrap procedure proposed by Wall and Stoffer (2002) requires the use of the backward representation. Furthermore, its implementation is complicated by the fact that the bootstrap density of the prediction errors is obtained in two steps. They obtain first the density that incorporates the parameter estimation uncertainty and then the density that takes into in account the variability of future innovations. Finally, these two densities are combined in the overall density of the prediction errors that is itself used to obtain the density of future observations. They show that their procedure works well in the context of univariate Gaussian state space models. However, it is computationally complicated in practice and it is difficult to extend it to more general models.

Next, we describe the procedure proposed by Wall and Stoffer (2002). The backward representation of state space models is based on the IF in equations (1.3a) and (1.4). To simplify the procedure, we consider that $d_t = c_t = 0$. Moreover, the system of matrices are assumed to be time-invariant, consequently, $P_{t|t-1} = \overline{P}$ and $F_t = \overline{F}$. Let's define $V_t^s = V_t\overline{F}^{-1/2}$, $t = 1, \ldots, T$, the standardized innovations. The following equations represent the backward recursion of the state space model in (1.1)

$$Y_t = N_t\tau_{t+1} - L_t a_{t|t-1} + M_t V_t^s, \quad t = T-1, \ldots, 1, \tag{1.14a}$$

$$\tau_t = T'\tau_{t+1} + A_t a_{t|t-1} - B_t V_t^s, \quad t = T-1, \ldots, 1, \tag{1.14b}$$

where τ_t is the reverse time estimate of the state vector with $\tau_T = \mathcal{V}_T^{-1}a_{T|T-1}$. The matrices in the backward recursions are given by $N_t = Z\mathcal{V}_t T' + \overline{FK}'$, $L_t = \overline{F}^{1/2}B_t' - ZV_tA_t$, $M_t = \overline{F}^{1/2}c_t - Z\mathcal{V}_tB_t$, $A_t = \mathcal{V}_t^{-1} - T'\mathcal{V}_{t+1}^{-1}T$, $B_t = T'\mathcal{V}_{t+1}^{-1}\overline{KF}^{1/2}$, $c_t = I - \overline{F}^{1/2}\overline{K}'\mathcal{V}_{t+1}^{-1}\overline{KF}^{1/2}$, $\mathcal{V}_{t+1} = T\mathcal{V}_tT' + \overline{KFK}'$ and $\overline{K} = T\overline{P}Z'$. These matrices are computed together with the forward Kalman filter with $\mathcal{V}_1 = \mathbb{E}\left[a_{1|0}a_{1|0}'\right]$.

Note that, as explained before, in practice the parameters are unknown and, consequently, the backward recursion in (1.14) should be carried out by substituting the unknown parameters by the corresponding QML estimates. In this case, the backward estimates of the state are denoted by $\widehat{\tau}_t$ for $t = 1, \ldots, T$.

The bootstrap prediction intervals of Y_{T+k} are obtained by the following steps:

Step 1: Estimate the parameters of model (1.1) by QML, $\widehat{\theta}$, and construct the standardized innovations $\left\{\widehat{V}_t^s; 1 \leq t \leq T\right\}$.

Step 2: Construct a sequence of bootstrap standardized innovations $\left\{\widehat{V}_t^{s*}; 1 \leq t \leq T+K\right\}$ via random draws with replacement from the standardized innovations, $\widehat{V}_t^s$, with $\widehat{V}_T^{*s} = \widehat{V}_T^s$.

Step 3: Construct a bootstrap replicate of the series, $\{Y_t^*; 1 \leq t \leq T-1\}$ via the backward state space model, in (1.14), with estimated parameters, $\theta = \widehat{\theta}$, using the innovations $\left\{\widehat{V}_t^{s*}; 1 \leq t \leq T-1\right\}$ and keeping $Y_T^* = Y_T$ fixed. Estimate the parameters of the model in order to obtain a bootstrap replicate, $\widehat{\theta}^*$, of them.

Step 4: Generate conditional forecasts $\left\{Y_{T+k|T}^*; 1 \leq k \leq K\right\}$ via the IF with estimated parameters and bootstrap errors as follows

$$a_{T+k|T}^* = \widehat{T}^k \widehat{a}_{T|T-1} + k\widehat{c} \sum_{j=0}^{k-1} \widehat{T}^{k-1-j} + \sum_{j=0}^{k-1} \widehat{T}^{k-1-j} \widehat{\widehat{K}\widehat{F}}^{-1} \widehat{V}_{T+j}^*, \tag{1.15a}$$

$$\begin{aligned} Y_{T+k|T}^* &= \widehat{Z}\widehat{T}^k \widehat{a}_{T|T-1} + \widehat{Z} \sum_{j=0}^{k-1} \widehat{T}^{k-1-j} + k\widehat{c}\widehat{Z} \sum_{j=0}^{k-1} \widehat{T}^{k-1-j} \\ &+ \widehat{\widehat{d}\widehat{K}\widehat{F}}^{-1} \widehat{V}_{T+j}^* + \widehat{V}_{T+k}^*, \; k = 1, 2, \ldots, \end{aligned} \tag{1.15b}$$

where $\widehat{V}_{T+j}^* = \widehat{\overline{F}}^{(1/2)} \widehat{V}_{T+j}^{s*}$

Step 5: Construct the conditional forecast values $\left\{\widehat{Y}_{T+k|T}^*; 1 \leq k \leq K\right\}$ via the IF with bootstrap parameters and future errors equal to zero, i.e.

$$\widehat{a}_{T+k|T}^* = \widehat{T}^{*k} \widehat{a}_{T|T-1} + k\widehat{c}^* \sum_{j=0}^{k-1} \widehat{T}^{*k-1-j} \tag{1.16a}$$

$$\widehat{Y}_{T+k|T}^* = \widehat{Z}^* \widehat{T}^{*k} \widehat{a}_{T|T-1} + k\widehat{c}^* \widehat{Z}^* \sum_{j=0}^{k-1} \widehat{T}^{*k-1-j} + \widehat{d}^*, \; k = 1, \ldots, . \tag{1.16b}$$

where $\widehat{a}_{T|T-1}^* = \widehat{a}_{T|T-1}$.

Step 6: Finally, compute the bootstrap forecast error by

$$d_k^* = Y_{T+k|T}^* - \widehat{Y}_{T+k|T}^*, \text{ for } k = 1, 2, \ldots, K \; .$$

Steps 2 to 6 are repeated B times.

Note that this procedure does not approximate directly the conditional distribution of Y_{T+k} but the distribution of the prediction errors. In Step 4 the bootstrap replicates

$Y^*_{T+k|T}$ are constructed using the estimated parameters. They incorporate the uncertainty due to the fact that when predicting, future innovations are equal to zero while in fact they are not. However these bootstrap replicates do not incorporate the uncertainty due to parameter estimation. Then, in Step 5 the bootstrap replicates $\widehat{Y}^*_{T+k|T}$ incorporate the variability attributable to parameter estimation through the use of $\widehat{\theta}^*$ instead of $\widehat{\theta}$. However, in $\widehat{Y}^*_{T+k|T}$, future innovations are assumed to be zero. Finally, the conditional bootstrap prediction errors, d^*_k, are computed as the difference between $Y^*_{T+k|T} - \widehat{Y}^*_{T+k|T}$. The corresponding prediction intervals, denoted by WS, are centered at the point prediction $\widetilde{Y}_{T+k|T}$. They are given by

$$\left[\widehat{Y}^{(i)}_{T+k|T} + Q^{*(i)}_{\alpha/2,d^*_k},\ \widehat{Y}^{(i)}_{T+k|T} + Q^{*(i)}_{1-\alpha/2,d^*_k}\right],\ i = 1,\ldots,N, \tag{1.17}$$

where $Q^{*(i)}_{\alpha/2,d^*_k}$ is the $\frac{\alpha}{2}$-percentile of the empirical conditional bootstrap distribution of the k-step ahead prediction errors of $Y^{(i)}_{T+k}$.

1.4 Organization of the thesis

In this thesis, we focus on two main problems. On one hand, in Chapter 2 we analyze the way in which the parameter uncertainty can affect the PMSE associated with the estimation of the underlying components and propose two new bootstrap procedures to obtain conditional PMSEs of the estimated unobserved states in state space models that incorporate the parameter uncertainty. We simplify the procedures with respect to alternative bootstrap PMSEs. The first bootstrap procedure proposed is parametric in the sense that it is based on constructing the bootstrap replicates by resampling the residuals from the assumed distribution of the errors with the estimated parameters. The second procedure is based on resampling from the empirical distribution of the residuals of the estimated model and consequently, it does not assume any particular distribution of the errors. We carry out Monte Carlo experiments to analyze the finite sample performance of our procedures. Finally, we implement the two new bootstrap procedures to a real macroeconomic problem and show how taking into account the parameter uncertainty can change some conclusion of interest for policy makers.

On the other hand, in Chapter 3, we propose bootstrap procedures to incorporate the uncertainty associated with parameter estimation into the prediction densities of future values of the observed series. The proposed procedure is not based on the backward representation of the model. Moreover, prediction densities of future observations are obtained in just one step. Consequently, our new bootstrap procedure has the advantage of being much simpler without loosing the already proven good small sample behavior of bootstrap procedures. In addition, from the computational point of view, our procedure is less time consuming. We carry out Monte Carlo experiments for analyzing the finite sample behavior of the new bootstrap procedure and compare them with alternatives. We also apply the new procedure to a real time series. We illustrate the performance of the proposed procedure to construct bootstrap prediction intervals by implementing it on the standardized quarterly mortgages change in home equity debt outstanding, unscheduled payments. We show that when the parameter uncertainty is not taken into account, the prediction intervals have a coverage that is under the nominal.

Finally, in Chapter 4, we summarize the main conclusions of this thesis and present some suggestions for future research.

Chapter 2

Bootstrap Prediction Mean Squared Errors of Unobserved States Based on the Kalman Filter with Estimated Parameters

2.1 Introduction

As mentioned in Chapter 1, the parameter uncertainty should be incorporated in the PMSEs of the estimated unobserved state when they are obtained by running the Kalman filter with estimated parameters. There are several alternatives proposed in the literature with this goal. They can be classified into three main groups. First, several proposals are based on the asymptotic distribution of the parameter estimator; see Chapter 1 for the description of the procedure proposed by Hamilton (1986). As we mentioned before, these procedures can be inadequate in small samples because, in this case, the asymptotic distribution could be a poor approximation of the finite sample distribution of the parameter estimator. Second, there are Bayesian procedures which provide the posterior distribution of the parameters conditional on the available information which is used for estimating the PMSE. However, these procedures can be very intensive computationally mainly when the structure behind the model is quite complicated; see for example Datta et al. (1999). Finally, in Chapter 1 we have also described the bootstrap procedures proposed by Pfeffermann and Tiller (2005) which have the advantage of being computationally simple even in relatively complicated models. The bootstrap PMSEs proposed by Pfeffermann and Tiller (2005) are based on obtaining the unconditional PMSEs of the estimates of the underlying states. However, one should note that the Kalman filter is designed to generate PMSEs conditional on the available information set. Although this distinction is irrelevant in state space models with time-invariant system matrices, it could be important when the system matrices in the state space model are observation dependent. Furthermore, by taking into account this distinction, it is possible to simplify the bootstrap procedures proposed by Pfeffermann and Tiller (2005) improving at the same time their finite sample performance.

In this chapter, we propose two new bootstrap procedures to obtain PMSEs of the Kalman filter estimates of the unobserved states in Gaussian state space models that incorporate the parameter uncertainty. By obtaining replicates of the underlying states conditional on the information available at each moment of time, we simplify the procedures with respect to alternative bootstrap PMSEs. The first bootstrap procedure proposed is parametric and it is based on resampling from the assumed distribution of the errors. Alternatively, we propose resampling from the residuals of the estimated model. Consequently, our proposed procedure does not assume any particular error distribution. We carry out Monte Carlo experiments to analyze the finite sample performance of our new procedures and compare them with that of the standard PMSE obtained from the Kalman filter with estimated parameters and with those of the asymptotic procedure of Hamilton (1986) and the bootstrap PMSEs proposed by Pfeffermann and Tiller (2005). We show that in small samples, the procedures proposed in this thesis have smaller biases than any of the other alternatives considered. We also show with simulated data that the proposed procedures can be implemented in the context of non-Gaussian models with good performance.

In this chapter, we also analyze how the parameter uncertainty can change the intervals of unobserved components in an empirical application. In particular, we considerer the estimation of the output gap, the non accelerating inflation rate of unemployment (NAIRU), the long-run investment rate and the core inflation in the US which are obviously variables of interest in the context of macroeconomic policy. We build on previous work by Doménech and Gómez (2006) who propose a multivariate unobserved components model for the US economy with the four unobserved variables mentioned above. They obtain prediction intervals of the unobserved output gap, NAIRU, core inflation and structural investment rate that do not incorporate the parameter uncertainty. We show that taking into account the additional uncertainty associated with the estimation of the parameters, the conclusions about the utility of the NAIRU as a macroeconomic indicator for expansions and recessions can be changed.

The rest of the chapter is organized as follows. In Section 2, we propose two new bootstrap procedures to obtain PMSEs of the one-step-ahead estimator of the unobserved states that take into account the parameter uncertainty. Section 3 analyzes the finite sample properties of the new procedures by means of Monte Carlo experiments, in the context of the random walk plus noise (RWN) model with homoscedastic, heteroscedastic and non-Gaussian errors. Section 4 contains the empirical application in which we estimate the uncertainty associated with the unobserved quarterly output gap, NAIRU, investment rate and core inflation in the US. Finally, Section 5 summarizes the main conclusions of the chapter.

2.2 A new bootstrap procedure

In this section, we propose two new bootstrap procedures to estimate the conditional PMSE of the one-step-ahead estimator of the unobserved components obtained by the Kalman filter run with estimated parameters. These procedures are similar to that proposed by Hamilton (1986) in the sense that we compute PMSE conditional on the available information set. However, instead of dealing with the parameter uncertainty by simulating the parameters from the asymptotic distribution of the corresponding estimator, we simulate them from a bootstrap distribution. In this way we obtain PMSEs with better small sample properties than those of Hamilton (1986). On the other hand, dealing with conditional PMSE allows us to simplify computationally the procedure with respect to the bootstrap procedures proposed by Pfeffermann and Tiller (2005) improving at the same time their performance in small samples. Furthermore, from an analytical point of view, the distinction between conditional and unconditional PMSEs can be important when dealing with models in which the system matrices are time-variant.

The first procedure proposed in this chapter is a parametric bootstrap procedure based on resampling from the assumed joint Gaussian distribution of the noises. Alternatively, we also propose a non-parametric procedure, based on resampling from the empirical distribution of the standardized one-step-ahead innovations, which does not assume any particular distribution of the errors.

Consider the state space model in equations (1.1) of Chapter 1. First, we describe the proposed parametric bootstrap algorithm to obtain the PMSE of $\widehat{a}_{t|t-1}$.

Step 1: Given the realization $\{Y_1, \ldots, Y_T\}$, estimate the parameters, $\widehat{\theta}$, and implement the Kalman filter to obtain the estimates of the underlying state, $\widehat{a}_{t|t-1}(\widehat{\theta})$, and the corresponding PMSE, $\widehat{P}_{t|t-1}(\widehat{\theta})$, $t = 1, \ldots, T$.

Step 2: Obtain a bootstrap replicate of the series $\{Y_1^*, \ldots, Y_T^*\}$ and of the underlying state $\{\alpha_1^*, \ldots, \alpha_T^*\}$, by extracting realizations, ε_t^* and η_t^*, $t = 1, \ldots, T$, from the joint Gaussian distribution of ε_t and η_t and using them in model (1.1) with the parameters substituted by $\widehat{\theta}$ as follows

$$\begin{aligned} \alpha_t^* &= \widehat{T}_t \alpha_{t-1}^* + \widehat{c}_t + \widehat{R}_{2t} \eta_t^*, \\ Y_t^* &= \widehat{Z}_t \alpha_t^* + \widehat{d}_t + \widehat{R}_{1t} \varepsilon_t^*, \end{aligned}$$

with $\alpha_0^* = \mathbb{E}(\alpha_t)$ or the initial observations for non-stationary processes. Estimate the bootstrap parameters, $\widehat{\theta}^*$.

Step 3: Run the Kalman filter with the original observations $\{Y_1, \ldots, Y_T\}$ and the bootstrap parameters estimated in step 2 as following

$$\begin{aligned} \widehat{a}_{t|t-1}(\widehat{\theta}^*) &= \widehat{T}_t a_{t-1|t-2}(\widehat{\theta}^*) + \widehat{c}_t \\ &\quad + K_t(\widehat{\theta}^*) F_{t-1}^{-1}(\widehat{\theta}^*) \left(Y_{t-1} - \widehat{d}_t - \widehat{Z}_t \widehat{a}_{t-1|t-2}(\widehat{\theta}^*) \right), \\ \widehat{P}_{t|t-1}(\widehat{\theta}^*) &= \widehat{T}_t \widehat{P}_{t-1|t-2}(\widehat{\theta}^*) \widehat{T}_t' - K_t(\widehat{\theta}^*) F_{t-1}^{-1}(\widehat{\theta}^*) K_t'(\widehat{\theta}^*) + \widehat{R}_{2t} \widehat{Q}_t \widehat{R}_{2t}^{*\prime}, \end{aligned}$$

where $\widehat{K}_t(\widehat{\theta}^*) = \widehat{T}_t \widehat{P}_{t-1|t-2}(\widehat{\theta}^*) \widehat{Z}_{t-1}'$, to obtain a bootstrap replicate of $\widehat{a}_{t|t-1}(\widehat{\theta}^*)$ and $\widehat{P}_{t|t-1}(\widehat{\theta}^*)$, $t = 1, \ldots, T$.

Steps 2 and 3 are repeated B times. Then, similarly as proposed by Hamilton (1986) in equation (1.7), the parametric conditional bootstrap PMSEs are obtained as follows

$$\begin{aligned} \widehat{PMSE}_{t|t-1}^{CB1} &= \frac{1}{B} \sum_{j=1}^{B} \widehat{P}_{t|t-1}^{(j)}(\widehat{\theta}^*) \\ &\quad + \frac{1}{B} \sum_{j=1}^{B} \left(\widehat{a}_{t|t-1}^{(j)}(\widehat{\theta}^*) - \widehat{a}_{t|t-1}(\widehat{\theta}) \right) \left(\widehat{a}_{t|t-1}^{(j)}(\widehat{\theta}^*) - \widehat{a}_{t|t-1}(\widehat{\theta}) \right)'. \end{aligned} \quad (2.1)$$

The first two steps are identical to those proposed by Pfeffermann and Tiller (2005); see section 1.2.2. However, in Step 3, we run the Kalman filter with the bootstrap estimates of the parameters and the original observations, while they run the filter with the bootstrap replicates of the series. In this way, we compute the PMSE conditional on the information contained in the original series, while the $\widehat{PMSE}_t^{PT}$ in equation (1.13) are unconditional. Furthermore, by computing the conditional PMSE, we avoid running the filter for each bootstrap replicate as it is done in Step 4 of Pfeffermann and Tiller (2005). This simplification implies a large reduction in computing time when estimating the PMSE of the underlying unobserved components.

We also propose a second non-parametric bootstrap procedure for estimating the conditional PMSE. Steps 1 and 3 are the same as in the parametric procedure just described. However, in Step 2, we construct the bootstrap replicates by resampling the standardized one-step-ahead innovations, $\widehat{V}_t^s$, and using the IF with the estimated parameters, $\widehat{\theta}$, as follows

$$a^*_{t+1|t} = \widehat{T}_{t+1}\widehat{a}^*_{t|t-1} + \widehat{c}_{t+1} + \widehat{K}^*_{t+1}\widehat{F}_t^{*-1}\widehat{V}_t^* \quad (2.2)$$

$$Y_t^* = \widehat{Z}_t\widehat{a}^*_{t|t-1} + \widehat{d}_t + \widehat{V}_t^*. \quad (2.3)$$

Then the bootstrap parameters, $\widehat{\theta}^*$ are estimated. Finally, the conditional PMSE is estimated as in equation (2.1) and is denoted by $\widehat{PMSE}_{t|t-1}^{CB2}$.

2.3 Monte Carlo experiments

In this section we carry out simulation experiments for evaluating the performance of the two new bootstrap procedures for estimating the conditional PMSEs and, compare their results with those of the standard PMSE given by the Kalman filter and the alternatives described in Chapter 1, namely, the asymptotic approximation proposed by Hamilton (1986) and the parametric bootstrap procedures proposed by Pfeffermann and Tiller (2005). We consider three Monte Carlo designs based on the RWN model with different assumptions about the distribution of its disturbances. In particular we consider a first model with homoscedastic Gaussian disturbances, a second model with heteroscedastic Gaussian disturbances and, finally, an homoscedastic model with non-Gaussian disturbances. The RWN model is defined by the following equations

$$y_t = \mu_t + \varepsilon_t \tag{2.4a}$$

$$\mu_t = \mu_{t-1} + \eta_t \tag{2.4b}$$

where y_t is the observation at time t of the series of interest and ε_t and η_t are mutually independent white noises with variances σ_ε^2 and $\sigma_\eta^2 = \sigma_\varepsilon^2 q$ respectively, where q is known as the signal-to-noise ratio. Although we consider this particular model for its simplicity, it has been successfully applied for explaining the dynamic evolution of many real time series; see, among many others, Commandeur and Koopman (2007) who fit the RWN to the log of the annual number of road traffic fatalities in Norway and Koopman and Bos (2004) and Stock and Watson (2007) who fit it for explaining the monthly US inflation.

2.3.1 Homoscedastic RWN model

We generate $R = 1000$ replicates of $\{y_t^{(j)}, \mu_t^{(j)}, j = 1, ..., R\}$ by model (2.4) with $\sigma_\varepsilon^2 = 1$ and $q = 0.25$, sample sizes $T = 40, 100$ and 500, and initial value equal to zero, $\mu_0 = 0$. For each replicate, we run the Kalman filter in (1.3) with known parameters to obtain one-step-ahead estimates of the underlying level, $\mu_t^{(j)}$, denoted by $m_{t|t-1}^{(j)}$ and their PMSE, denoted by $P_{t|t-1}^{(j)}$. Furthermore, for each simulated series j and moment of time t, we also generate 10000 replicates of $\mu_{t+1}^{(j)}$, denoted by $\mu_{t+1}^{(j,i)}$, $i = 1, ..., 10000$, from the corresponding conditional distribution in (1.2). Then, at each moment of time, we compute the empirical conditional PMSE of $m_{t|t-1}^{(j)}$ given by $PMSE_{t|t-1}^{(j)} =$

$\frac{1}{10000}\sum_{i=1}^{10000}\left(\mu_t^{(j,i)} - m_{t|t-1}^{(j)}\right)^2$ and the relative bias $d_t^{(j)} = P_{t|t-1}^{(j)}/PMSE_{t|t-1}^{(j)} - 1$. Moreover, in order to compare the two new bootstrap procedures with the parametric bootstrap procedure proposed by Pfeffermann and Tiller (2005), we also compute the empirical unconditional PMSE of $m_{t|t-1}^{(j)}$ which is given by $UPMSE_t = \frac{1}{R}\sum_{j=1}^{R}\left(\mu_t^{(j)} - m_{t|t-1}^{(j)}\right)^2$ and the corresponding relative bias $ud_t^{(j)} = P_{t|t-1}^{(j)}/UPMSE_{t|t-1}^{(j)} - 1$. For each replicate, we estimate the parameters by QML using as starting values for the filter $\widehat{m}_{1|0} = 0$ and $\widehat{P}_{1|0} = \infty$. Then, as before, we calculate the empirical conditional and unconditional PMSE of $\widehat{m}_{t|t-1}$ given by $PMSE_{t|t-1}^{(j)} = \frac{1}{10000}\sum_{i=1}^{10000}\left(\mu_t^{(j,i)} - \widehat{m}_{t|t-1}^{(j)}\right)^2$ and $UPMSE_{t|t-1}^{(j)} = \frac{1}{R}\sum_{i=1}^{R}\left(\mu_t^{(j)} - \widehat{m}_{t|t-1}^{(j)}\right)^2$ respectively, and their corresponding relative bias d_t^j and ud_t^j. On the other hand, for each Monte Carlo replicate, we also generate $M = 1000$ realizations of the parameters, σ_ε^2 and q, from the asymptotic distribution of the QML estimator to obtain the $\text{PMSE}_{t|t-1}^{Asy}$ proposed by Hamilton (1986) in equation (1.7). Finally, for each Monte Carlo replicate, we generate $B = 1000$ bootstrap replicates and obtain the parametric bootstrap $\text{PMSE}_{t|t-1}^{PT}$ proposed by Pfeffermann and Tiller (2005) in equation (1.13), and the two new bootstrap procedures proposed in this thesis.

Figure 2.1 plots the averages of the relative biases $d_t^{(j)}$ of the conditional PMSE over the Monte Carlo replicates for all procedures considered in this thesis[1]. When the parameters are known the relative biases are denoted as KF1 which, as expected, evolve

[1]Note that when the Kalman filter is run the effect of the initial values on the estimates of the PMSE vanishes in approximately five iterations; see Ray (1989). Consequently, we remove $\widehat{P}_{t|t-1}$, for $t = 1$ to 5, for calculating the corresponding biases $d_t^{(j)}$.

Table 2.1: Averages and standard deviations (Std) through time of the relative biases (in percentage) of PMSE of the underlying level in the RWN models with Gaussian homoscedastic, Gaussian heteroscedastic and non-Gaussian errors.

	Homoscedastic		Heteroscedastic		Non-Gaussian	
	Average	Std	Average	Std	Average	Std
			T = 40			
KF 1[a]	0.02	0.04	0.02	0.04	-2.69	9.84
KF 2[b]	-8.02	0.57	-15.44	0.71	-10.85	6.81
Asy[c]	20.53	15.34	20.71	1.23	21.59	19.30
PT	-7.62	0.68	-11.72	1.12	-94.62	8.58
CB 1	-1.46	0.61	-1.63	1.28	-3.25	3.75
CB 2	-1.21	0.60	-1.87	1.28	-3.42	3.54
			T = 100			
KF 1	0.02	0.01	0.02	0.04	-1.18	6.52
KF 2	-6.82	0.25	-6.24	0.24	-7.02	4.92
Asy	-3.88	0.22	11.03	0.73	-4.32	4.97
PT	-3.55	0.20	-3.20	0.50	-96.53	6.08
CB 1	-0.64	0.37	-0.79	0.36	-2.12	2.60
CB 2	-0.56	0.37	-2.33	0.36	-2.35	2.58
			T = 500			
KF 1	0.02	0.05	0.02	0.05	-0.37	3.18
KF 2	-0.97	0.14	-1.25	0.10	-1.79	2.74
Asy	-0.29	0.15	-0.23	0.11	-1.65	2.74
PT	0.20	0.15	-0.80	0.34	-99.10	2.35
CB 1	-0.18	0.15	-0.96	0.12	-1.86	1.59
CB 2	-0.25	0.15	-1.08	0.12	-1.60	1.63

[a] Kalman filter procedure with known parameters.
[b] Kalman filter procedure with estimated parameters.
[c] Asymptotic approximation proposed by Hamilton (1986).

around zero through time regardless of the sample size considered. The average and standard deviations through time of the Monte Carlo averages plotted in Figure 2.1 have been reported in the first two columns of Table 2.1. Note that the average relative biases and their standard deviations, which as expected are very small, do not depend on the sample size. These biases and standard deviations can be attributed to the simulation error. When the parameters are substituted by their QML estimates, the

Table 2.2: Averages and standard deviations (Std) through time of the relative biases (in percentage) of unconditional PMSE of the underlying level in the RWN models with Gaussian homoscedastic, Gaussian heteroscedastic and non-Gaussian errors.

	Homoscedastic		Heteroscedastic		Non-Gaussian	
	Average	Std	Average	Std	Average	Std
			T = 40			
KF 1[a]	-0.51	5.19	-1.25	4.58	-2.11	5.68
KF 2[b]	-9.62	4.72	-10.33	3.94	-5.62	5.23
Asy[c]	27.92	3.42	35.92	2.78	21.59	19.30
PT	1.12	3.37	-0.60	2.77	-94.62	8.58
CB 1	2.76	3.61	-1.71	2.76	-3.25	3.75
CB 2	2.51	3.62	-1.96	2.76	-3.42	3.54
			T = 100			
KF 1	0.25	5.07	1.51	4.19	1.00	7.21
KF 2	-2.20	4.69	-2.67	3.83	-3.55	6.68
Asy	-1.16	3.28	17.30	2.61	-5.46	4.97
PT	0.11	3.28	2.27	2.65	-96.53	6.08
CB 1	-1.10	3.29	4.82	2.61	-2.12	2.60
CB 2	-1.09	3.29	3.19	2.61	-2.35	2.58
			T = 500			
KF 1	-0.13	4.34	1.04	4.50	0.68	5.66
KF 2	0.22	3.89	-0.01	3.84	-1.18	5.60
Asy	0.26	2.79	1.08	2.74	-1.87	2.74
PT	0.74	2.79	1.22	2.77	-99.10	2.35
CB 1	0.23	2.79	0.24	2.74	-1.16	1.59
CB 2	0.10	2.79	0.12	2.74	-1.26	1.63

[a] Kalman filter procedure with known parameters.
[b] Kalman filter procedure with estimated parameters.
[c] Asymptotic approximation proposed by Hamilton (1986).

relative biases through time, denoted as KF2, evolve around approximately 9% when $T = 40$. Obviously, because this bias is caused by using estimated parameters which are obtained by consistent estimators, it disappears as the sample size increases. The average biases reported in Table 2.1 show that when the Kalman filter is run with estimated parameters $\widehat{P}_{t|t-1}$ is a negatively biased estimator of the conditional true

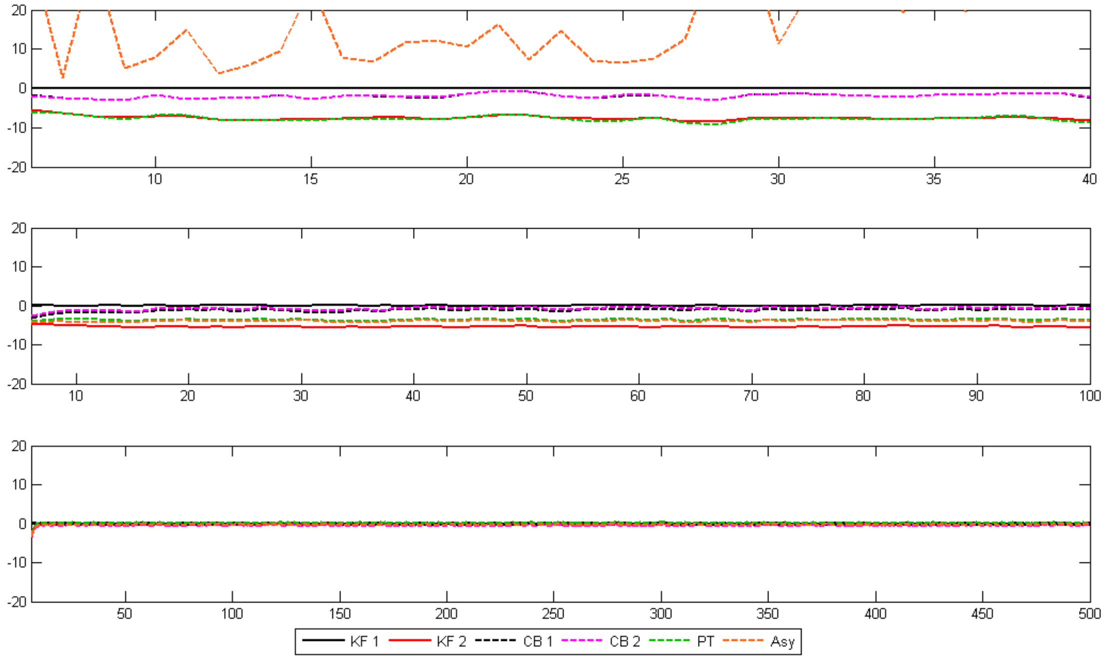


Figure 2.1: Monte Carlo averages of the ratios $d_t = 100 \times \left(\frac{P_{t|t-1}}{PMSE_t} - 1\right)$ for the RWN model with homoscedastic Gaussian error and $T = 40$ (first row), $T = 100$ (second row) and $T = 500$ (third row).

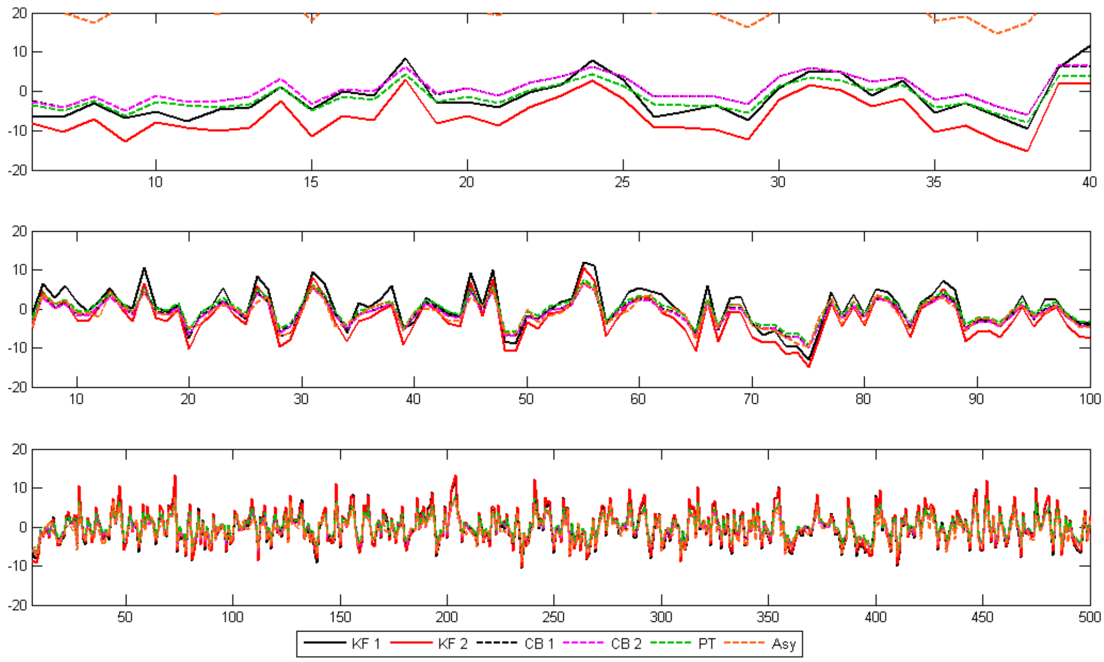


Figure 2.2: Monte Carlo averages of the ratios $ud_t = 100 \times \left(\frac{P_{t|t-1}}{UPMSE_t} - 1\right)$ for the RWN model with homoscedastic Gaussian error and $T = 40$ (first row), $T = 100$ (second row) and $T = 500$ (third row).

PMSE of $\widehat{m}_{t|t-1}$. The biases are as large as -8% when $T = 40$ and -6.82% when $T = 100$. On the other hand, Figure 2.2 shows the same information plotted in Figure 2.1, but for the biases corresponding to the unconditional PMSE, $du_t^{(j)}$. The corresponding averages and the standard deviations through time of the relative biases are reported in Table 2.2. When the parameter are known, the relative biases evolve around zero. However, they are larger and much volatile than those of the conditional PMSE, even for large sample size. When the parameters are substituted by their corresponding QML estimates, the relative biases are -9.62% when $T = 40$, -2.2% when $T = 100$ and, 0.22% when $T = 500$. Consequently, $\widehat{P}_{t|t-1}$ is a negatively biased estimator of the unconditional true PMSE of $\widehat{m}_{t|t-1}$.

Figure 2.1 also plots the relative biases of the asymptotic estimator of the conditional PMSE in (1.7), denoted as Asy, for the RWN model considered above. When $T = 40$ the biases are even larger in absolute value than when the PMSE are computed with estimated parameters. Table 2.1, that reports the averages through time, shows that this quantity for the asymptotic procedure is around 20%, while the relative bias of the PMSEs obtained from the Kalman filter with estimated parameters is -8%. This result illustrates in small samples that, the asymptotic distribution can be a very inadequate approximation of the sample distribution of the QML estimator in unobserved component models. Obviously, as the QML estimator is consistent, the biases decrease with the sample size. Similar results occur when the unconditional PMSE is measured. When $T = 40$ the biases of the PMSE proposed by Hamilton (1986) is almost out of the range of Figure 2.2. In particular, Table 2.2 shows that it is about 28%. But, as in the case of the conditional PMSEs, the relative biases decrease when the sample size increases.

Figure 2.1 also plots the Monte Carlo averages of the relative biases of the parametric bootstrap procedure proposed by Pfeffermann and Tiller (2005) and denoted by PT. For this procedure, when $T = 40$, the relative biases of $\text{PMSE}^{PT}_{t|t-1}$ are smaller than those of the asymptotic PMSEs, but only slightly smaller than the biases obtained when the Kalman filter is run with estimated parameters. This fact is clearly observed in Table 2.1 where the bias of the parametric bootstrap PMSE is -7.62% compared with -8.02% in the Kalman filter with estimated parameters. In the moderate sample size, $T = 100$, the procedures proposed by Hamilton (1986) and Pfeffermann and Tiller (2005) have similar relative biases. Finally, when $T = 500$, the parametric bootstrap PMSEs are approximately unbiased.

Finally, consider the two new bootstrap procedures proposed in this chapter. When the proposed parametric and non-parametric bootstrap PMSE, denoted by CB1 and CB2 respectively, are used for estimating the conditional PMSE, Figure 2.1 shows that, regardless of the sample size, the biases of these procedures are very similar. These biases obviously decrease with the sample size and are clearly smaller than those observed when the PMSEs are computed using the Kalman filter with estimated parameters, the asymptotic PMSEs proposed by Hamilton (1986) or the parametric bootstrap procedure proposed by Pfeffermann and Tiller (2005) when the sample sizes are small or moderate. The time averages and standard deviations reported in Table 2.1 show that the reductions of the relative biases can be very important when $T = 40$. For example, the relative bias is -8.02% when using the Kalman filter with estimated parameters, 20.53% when using the asymptotic distribution of the QML estimator, -7.62% when using the bootstrap procedure of Pfeffermann and Tiller (2005) while they are as small as -1.46% and -1.21% when using the parametric and non-parametric bootstrap procedures proposed in this chapter. The reduction of the relative biases is still important when $T = 100$ while when $T = 500$ all procedures to compute the conditional PMSEs of the estimates of the unobserved level μ_t are approximately unbiased. The good performance of the parametric bootstrap procedures could be expected given that the model is conditionally Gaussian and in the parametric procedure, we are resampling from the true Gaussian distribution. However, it is comforting to observe that the behavior of the non-parametric procedure which does not assume any particular distribution is comparable with that of the parametric procedure.

When the unconditional PMSE is considered, it can be observed in Table 2.2 that, the procedure for estimating the PMSE proposed by Pfeffermann and Tiller (2005) perform better for in small sample, $T = 40$, its relative bias is 1.12%, while for both the CB1 and CB2 bootstrap procedures are 2.76% and 2.51% respectively. However, the relatives biases are significantly smaller for the CB1 and CB2 procedures than those of the PMSE obtained with the estimated parameters and the asymptotic approximation proposed by Hamilton (1986). Moreover, these biases seems to disappear when the sample size increases. Figure 2.2 confirms all the results in Table 2.2, specially those for small sample size.

2.3.2 Heteroscedastic RWN model

In this subsection we carry out simulations generating replicates by a time-varying state space model. In particular, we consider the RWN model in (2.4) where the transitory noise, ε_t, is heteroscedastic and given by

$$\varepsilon_t = \varepsilon^{\dagger}\sigma_t, \tag{2.5}$$

where $\sigma_t^2 = \alpha_0 + \alpha_1 v_{t-1}^2$, v_t is the one-step-ahead innovation given by $v_t = y_t - m_{t|t-1}$, and $\varepsilon_t^{\dagger}$ is a Gaussian white noise process with variance 1, distributed independently of η_t.

Note that, given the specification of σ_t^2 and assuming that the parameters are known, the model is still conditionally Gaussian since knowledge of past observations and past estimates of the state implies knowledge of past innovations v_{t-1}. This model is related with the STARCH model described by Harvey et al. (1992) but they differ in that the STARCH model assumes that $\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$ and, consequently, it is not conditionally Gaussian. Unobserved component models with heteroscedastic disturbances are becoming very popular to represent the dynamic evolution of macroeconomic variables as, for example, inflation or electricity prices; see, Broto and Ruiz (2009), Jungbacker et al. (2009) and Stock and Watson (2007) among many others. In this case, we also generate $R = 1000$ series from the heteroscedastic RWN model with ε_t defined as in (2.5) with $\alpha_0 = 0.6719$, $\alpha_1 = 0.2$ and $\sigma_\eta^2 = 0.25$. The initial conditions are given by $\mu_0 = 0$ and σ_1^2 equal to the marginal variance of ε_t, which is one. As before, for each simulated series j and moment of time t, we also generate 10000 replicates of $\mu_{t+1}^{(j)}$, denoted by $\mu_{t+1}^{(j,i)}$, $i = 1, ..., 10000$, from the corresponding conditional distribution in (1.2), then we run the Kalman filter with known parameters for each simulated series and compute $m_{t|t-1}^{(j)}$ and $P_{t|t-1}^{(j)}$ and the corresponding relative biases $d_t^{(j)}$ and $ud_t^{(j)}$.

Once more, when the Kalman filter is run with known parameters, the relative biases of the conditional PMSE, denoted as KF1 in Figure 2.3, evolve around zero for all sample sizes. This is also observed in Table 2.1, that reports the averages through time of the relative biases which are 0.02% for the three sample sizes. However, when the estimation of the unconditional PMSE is considered, the relative biases of the PMSEs computed with known parameters are larger than those for the conditional PMSEs. In

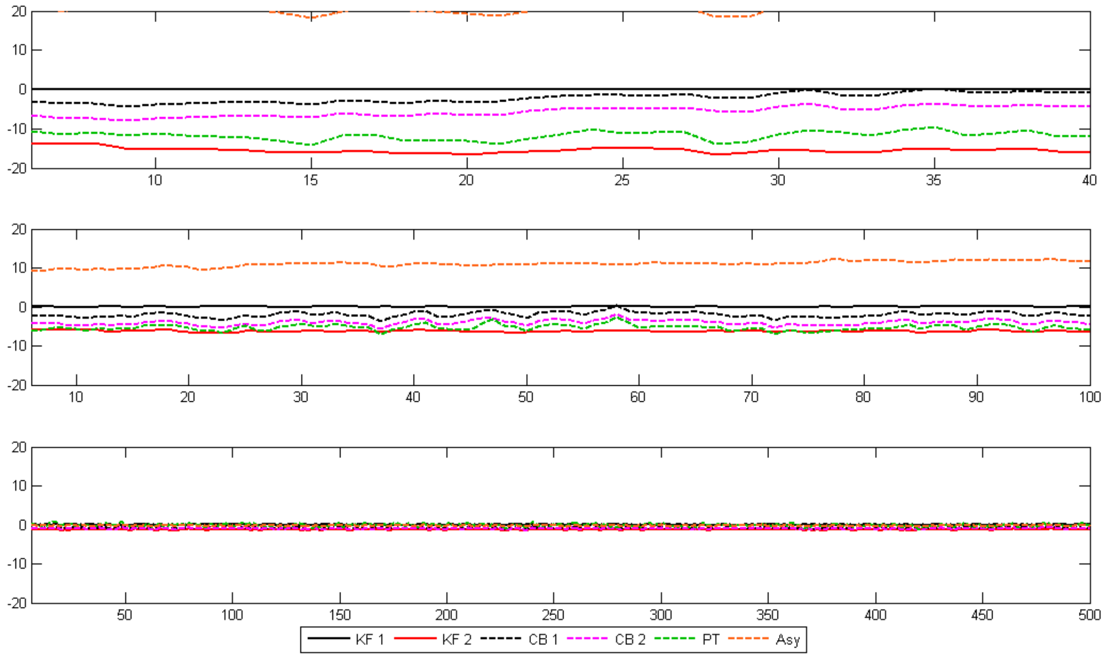


Figure 2.3: Monte Carlo averages of the ratios $d_t = 100 \times \left(\frac{P_{t|t-1}}{PMSE_t} - 1\right)$ for the RWN model with heteroscedastic Gaussian error and $T = 40$ (first row), $T = 100$ (second row) and $T = 500$ (third row).

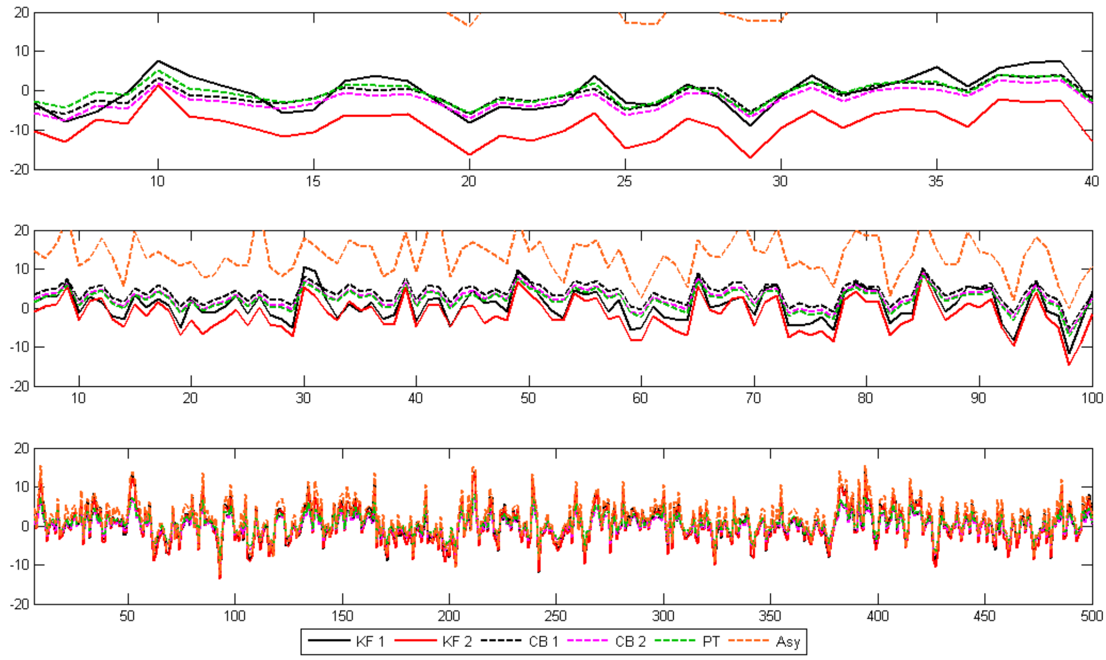


Figure 2.4: Monte Carlo averages of the ratios $ud_t = 100 \times \left(\frac{P_{t|t-1}}{UPMSE_t} - 1\right)$ for the RWN model with homoscedastic Gaussian error and $T = 40$ (first row), $T = 100$ (second row) and $T = 500$ (third row).

particular, Table 2.2 reports that the relative biases are -1.25% when $T = 40$, 1.51% when $T = 100$ and 1.04% when $T = 500$. Consequently, it not clear that the relative, $ud_t^{(j)}$, bias decreases with the sample size. Note that in this case, the model is not time-invariant and, consequently, the conditional and unconditional PMSE do not coincide. Given that the Kalman filter is designed to obtain the conditional PMS, it could be expected that it gives biased unconditional PMSE even when the sample sizes are large.

When the Kalman filter is run with estimated parameters, Figure 2.3 plots the corresponding average biases denoted by KF2 which, similarly to those plotted in Figure 2.1 for the time-invariant model, are clearly below of zero for the small and moderate sample sizes. The central columns of Table 2.1 report the averages and standard deviations through time of the averages of the biases $d_t^{(j)}$, plotted in Figure 2.3 which behave similar to those reported in the time-invariant case, except for the small sample size. In this latter case, the average relative bias is -15.44% compared with -8.02% in the time-invariant model. As expected, these relative biases tend to decrease when the sample size increases.

Figure 2.4, that reports the averages through the Monte Carlo replicates of the relative biases of the unconditional PMSE, $ud_t^{(j)}$, shows that when $T = 40$ or $T = 100$, the relative biases of the KF2 are negative bias and larger than those of the conditional PMSE.

Figure 2.3 also plots the relative biases of the asymptotic estimator of the conditional PMSE in (1.7). When $T = 40$ the biases are even larger than when the PMSE are computed with the Kalman filter with estimated parameters. Table 2.1 shows that the relative bias is around 20% compared with that of the Kalman filter with estimated parameters which is -15%. Obviously, as the QML estimator is consistent, the biases decrease with the sample size. However, it is important to note that even when $T = 100$ the biases are larger than for the time-invariant model. With respect to the unconditional PMSE, Table 2.2 shows that when $T = 40$ and $T = 100$ the relative biases are very large, 35.92% and 17.30% respectively. This result is clearly observed in Figure 2.4 where the relative biases associated with the Asy PMSEs are far from the zero line.

Finally, with respect to the two new bootstrap procedures, Figure 2.1 shows that, as in the time-invariant model, the biases of the proposed parametric and non-parametric bootstrap PMSE are very similar. These biases decrease with the sample size and are clearly smaller than those observed when the conditional PMSEs are computed using the Kalman filter with estimated parameters, the asymptotic proposal of Hamilton (1986) or the parametric bootstrap procedure proposed by Pfeffermann and Tiller (2005) when the sample sizes are small or moderate. The time averages and standard deviations reported in Table 2.1 show that the reductions of the relative biases can be very important when $T = 40$. For example, the relative bias is -15.44% when using the Kalman filter with estimated parameters, 20.71% when using the asymptotic distribution of the QML estimator, -11.72% when using the bootstrap procedure of Pfeffermann and Tiller (2005) while for the two new bootstrap procedures the relative biases are as small as -1.63% and -1.87% when using the parametric and non-parametric procedures proposed in this thesis respectively. The reduction of the relative biases is still important when $T = 100$, while when $T = 500$ all procedures to compute the conditional PMSEs of the estimates of the unobserved level μ_t are approximately unbiased. It is also remarkable that the relative biases and standard deviations of the parametric and non-parametric bootstrap procedures proposed in this chapter are approximately the same.

Therefore, our simulation results show that in small and moderate sample sizes the proposed bootstrap procedures to compute the conditional PMSE of $\widehat{a}_{t|t-1}$ have very small biases which are smaller than those of alternative procedures. Furthermore, this reduction of bias is accomplished BY using procedures which are simpler from a computational point of view. It is also important to point out that we have considered a very simple model in order to illustrate the performance of the CB1 and CB2 procedures. Therefore, it is expected that the simplicity of our procedures when compared with alternatives is going to be even more important when dealing with more complicated models.

2.3.3 Non-Gaussian RWN model

Remember that when the conditional Normality assumption is not satisfied, the Kalman filter in equations (1.3a) and (1.3b) do not provide the conditional mean of the unobserved states and their corresponding conditional PMSEs. However, $a_{t|t-1}$ still are optimal one-step-ahead estimates of the underlying state in the sense that they have minimum PMSE, given by $P_{t|t-1}$, among all estimators which are linear functions of the observations. Taking into account this feature, in this section, we analyze the robustness of the two new bootstrap procedures proposed in this chapter to estimate the conditional and unconditional PMSE of the unobserved components in non-Gaussian unobserved component models. We focus on a particular specification of interest in the context of Stochastic Volatility models. In particular, we consider again the RWN model in (2.4) with the measurement equation disturbance, ε_t, having a $\log(\chi_1^2)$ distribution; see, for instance, Harvey et al. (1994) for the relation between this model and the linear transformation of the Autoregressive Stochastic Volatility Model. In order to guarantee that the variances of the two error terms are equal to those of the homoscedastic Gaussian model considered in the previous sections, we center and re-scaled the $\log(\chi_1^2)$. In addition, given that the model is not conditionally Gaussian, the distribution in (1.2) is not further the true conditional distribution of the vector $(\alpha_t, Y_t)'$. Consequently, for each simulated series j and moment of time t, we generate 10000 replicates of $\mu_t^{(j)}$, $\mu_t^{(j,i)}$, $i = 1, \ldots, 10000$, by particle filtering; see, Kitagawa (1996) and Arulampalam et al. (2002) for details about particle filtering procedures. Then, the empirical conditional and unconditional PMSEs and their corresponding relative biases are computed as in previous sections.

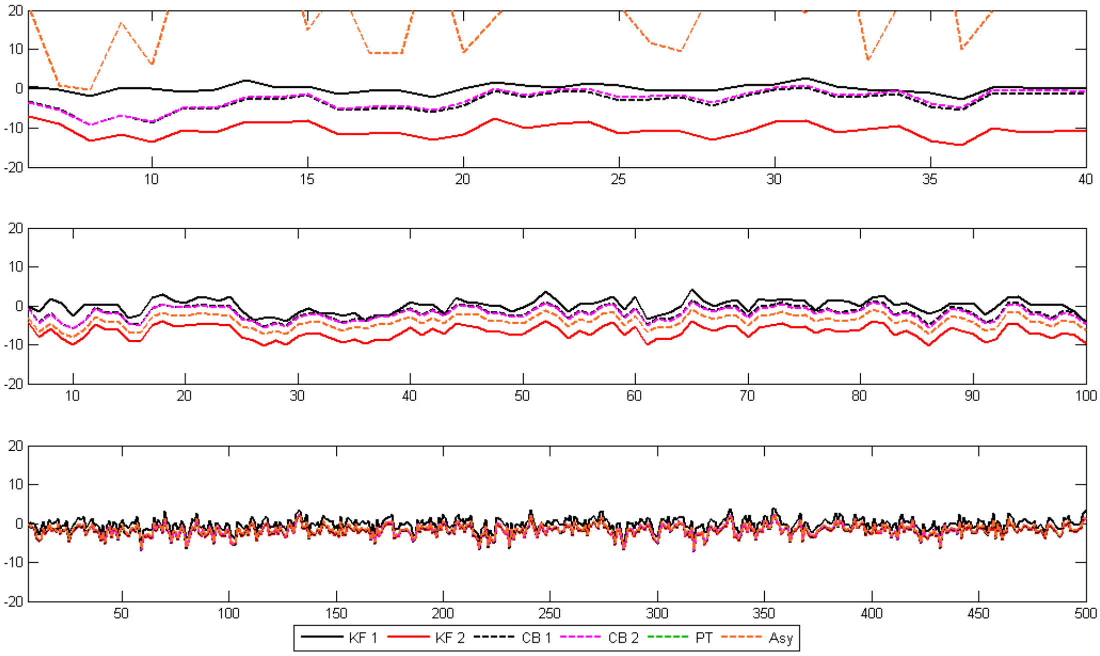


Figure 2.5: Monte Carlo averages of the ratios $d_t = 100 \times \left(\frac{P_{t|t-1}}{PMSE_t} - 1\right)$ the RWN model with error term ε_t distributed as $\log \chi_1^2$ and $T = 40$ (first row), $T = 100$ (second row) and $T = 500$ (third row).

Figure 2.5 plots the averages through Monte Carlo replicates of the relative biases when the Kalman filter is run with known parameters. First of all this figure illustrates that

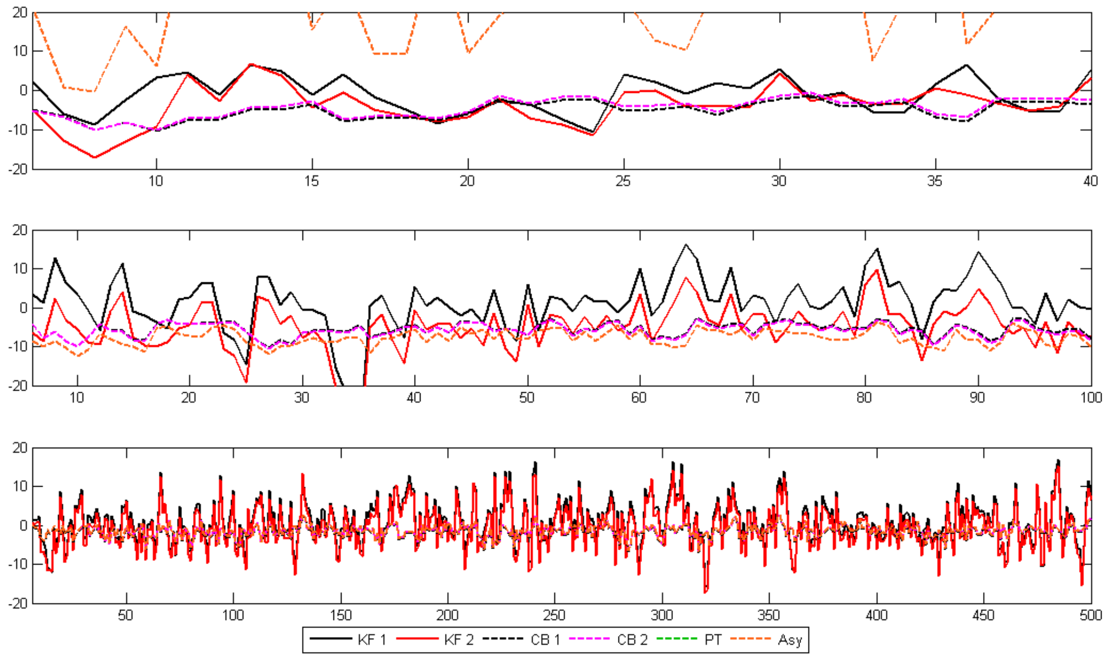


Figure 2.6: Monte Carlo averages of the ratios $ud_t = 100 \times \left(\frac{P_{t|t-1}}{UPMSE_t} - 1\right)$ for the RWN model with homoscedastic Gaussian error and $T = 40$ (first row), $T = 100$ (second row) and $T = 500$ (third row).

even in this case, $P_{t|t-1}$ are slightly biased estimates of the true conditional PMSE. This result can also be observed in the fifth column of Table 2.1 that reports the averages and standard deviations through time of the relative biases plotted in Figure 2.5. In the small sample, when $T = 40$, the relative bias is -2.69% and this bias decrease with the sample size. However, the standard deviations are much larger than those reported for the conditional Gaussian models. These biases can be attributed to the fact that when the model is not conditionally Gaussian, $m_{t|t-1}$ is not the true conditional mean of μ_t. When the Kalman filter is run with known parameters for estimating the unconditional PMSE of the estimates of the unobserved components, the averages reported in Table 2.2 are similar in absolute value to those reported in Table 2.1 for the estimation of the conditional PMSE. In particular, the relative biases for the unconditional PMSE are -2.11%, 1.00% and 0.68% for $T = 40, 100$ and 500 respectively, while for the conditional PMSE they are -2.69%, -1.18% and -0.37% for

$T = 40, 100$ and 500 respectively. Moreover, their corresponding standard deviations are also similar; see also Figure 2.6.

On the other hand, the relative biases for estimating the conditional PMSE reported in the fifth column of Table 2.1 for the PMSE computed with the Kalman filter with estimated parameters, which have been estimated by QML by maximizing the Gaussian log-likelihood, are not very different from those reported for the conditional Gaussian models. However, once more, the standard deviations are much larger.

The result observed for the asymptotic procedure proposed by Hamilton (1986) seems to be robust to the presence of non-Gaussianity. However, as it is observed in Table 2.1, the relative biases are larger than those for the Kalman filter with known and estimated parameters when $T = 40$. On the other hand, it is remarkable that, the parametric bootstrap procedure proposed by Pfeffermann and Tiller (2005) has very large biases which do not decrease with the sample size for both the conditional and unconditional PMSEs. The parametric bootstrap are based in resampling from the centered and re-scaled $\log\left(\chi_1^2\right)$ distribution to obtain replicates of ε_t while the replicates of η_t are obtained by resampling from a $\mathcal{N}\left(0, \widehat{\sigma}_\eta^2\right)$ distribution. It has in average an overestimation of the true PMSE of approximately 95% for all sample sizes.

Finally, the two new bootstrap procedures proposed in this chapter have an adequate performance even in the small sample size. In particular, the relative biases of the new parametric and non-parametric procedures for estimating the conditional PMSE are -3.25% and -3.42%, respectively when $T = 40$, and, -2.12% and -2.35% when $T = 100$. They are clearly smaller than the biases of any of the three alternative feasible estimators of the PMSE and very close to those reported for the PMSE computed by the Kalman filter with known parameters. Finally, notice that as in the Gaussian

RWN models, the biases and standard deviations of the parametric and non-parametric procedures are very similar.

Therefore, we can conclude that the proposed procedures can be implemented in non-Gaussian state space models with adequate performances.

2.4 Empirical application: Estimating the Output Gap, NAIRU, Trend Investment Rate and Core Inflation in US

In this section, we apply the proposed bootstrap estimators of the PMSE of the unobserved components to estimate the uncertainty associated with the estimation of the output gap of the US economy based on the unobserved components model proposed by Doménech and Gómez (2006). Remember that, as mentioned in the introduction, this model is multivariate. Therefore, as a by product, we illustrate how the bootstrap procedures proposed in this chapter to estimate PMSE of the unobserved components can also be implemented in multivariate systems. Furthermore, further to the output gap, the model also has several other unobserved variables, as the NAIRU, the investment trend and the core inflation. We compute the PMSE for each of the four unobserved variables in the model. First, we obtain them by running the Kalman filter without taking into account the parameter uncertainty. We also compute the PMSE using the asymptotic approximation of Hamilton (1986) and the parametric bootstrap procedure of Pfeffermann and Tiller (2005). Finally, we implement the parametric and non-parametric procedures proposed in this chapter.

The model proposed by Doménech and Gómez (2006) has the following unobserved component form

$$\begin{aligned}
y_t &\equiv y_t^p + z_t, & (2.6a)\\
z_{t+1} &= 2\theta_1 \cos\theta_2 z_{t-1} - \theta_1^2 z_{t-2} + \omega_{zt}, & (2.6b)\\
y_{t+1}^p &= \overline{\mu} + y_t^p + \omega_{yt}, & (2.6c)\\
\pi_t &= \left(1 - \sum_{i=1}^{4} \mu_i\right)\overline{\pi}_t + \left(\sum_{i=1}^{4} \mu_i \pi_{t-i}\right) + \eta_y z_t + v_{\pi t}, & (2.6d)\\
\overline{\pi}_t &= \overline{\pi}_{t-1} + \omega_{\pi t}, & (2.6e)\\
U_t &= \phi_u U_{t-1} + (1-\phi_u)\overline{U}_t + \phi_0 z_t + v_{ut}, & (2.6f)\\
\overline{U}_t &= \overline{U}_{t-1} + \omega_{ut}, & (2.6g)\\
x_t &= \beta_x x_{t-1} + (1-\beta_x)\overline{x}_t + \beta_{y_0} z_t + \beta_{y_1} z_{t-1} + v_{xt}, & (2.6h)\\
\overline{x}_t &= \overline{x}_{t-1} + \omega_{xt}, & (2.6i)
\end{aligned}$$

where y_t is the logarithm of the GDP, z_t is the unobserved output gap which is assumed to follow a cyclical AR(2) process in equation (2.6b) and, y_t^p is the logarithm of the potential output represented by a random walk plus drift model in equation (2.6c). $\overline{\mu}$ captures the growth rate of the potential output. The noises ω_{zt} and ω_{yt} are assumed to be mutually independent Gaussian white noises with zero mean and variances $\sigma_{\omega z}^2$ and $\sigma_{\omega y}^2$ respectively. The following two equations, (2.6d) and (2.6e), describe the dynamic evolution of inflation, π_t and its relation with the output gap. $\overline{\pi}_t$ is the core inflation which follows a random walk. The noises $v_{\pi t}$ and $\omega_{\pi t}$ are Gaussian white noises with variances $\sigma_{v\pi}^2$ and $\sigma_{\omega\pi}^2$ respectively. Both noises are mutually independent and independent of ω_{zt} and ω_{yt}. Equations, (2.6f) and (2.6g) describe the Okun's law where U_t is the unemployment rate and $\overline{U}_t$ is the NAIRU. Once more, the disturbances associated with the unemployment, v_{ut} and ω_{ut} are Gaussian white noises with variances σ_{vu}^2 and $\sigma_{\omega u}^2$, respectively. They are mutually independent and independent of the rest of disturbances in the model. Finally, the last two equations, (2.6h) and (2.6i) describe the dynamic evolution of the investment rate, x_t defined as $x_t \equiv investment_t/output_t$, and of the long run investment trend $\overline{x}_t$. The disturbances v_{xt} and ω_{xt} are Gaussian white noises with zero mean variances σ^2_{vx} and $\sigma^2_{\omega x}$ and, once more, they are assumed to be mutually independent and independent of all previous disturbances.

Model (2.6) can be casted into a state space framework as in model (1.1) with $Y_t = \left[y_t, U_t - \phi_u U_{t-1}, x_t - \beta_x x_{t-1}, \pi_t - \left(\sum_{i=1}^4 \mu_i \pi_{t-i}\right)\right]'$, $\alpha_t = \left[y_t^p, \overline{U}_t, \overline{x}_t, \overline{\pi}_t, z_{t-2}, z_{t-1}, z_t\right]'$, $\varepsilon_t = [v_{ut}, v_{xt}, v_{\pi t}]$, $\eta_t = [\omega_{yt}, \omega_{ut}, \omega_{xt}, \omega_{\pi t}, \omega_{zt}]$, $H_t = diag\left\{\sigma^2_{vu}, \sigma^2_{vx}, \sigma^2_{v\pi}\right\}$, $Q_t = diag\left\{\sigma^2_{\omega y}, \sigma^2_{\omega u}, \sigma^2_{\omega x}, \sigma^2_{\omega \pi}, \sigma^2_{\omega z}\right\}$,

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -\theta_1^2 & 2\theta_1 \cos\theta_2 \end{bmatrix},$$

$$Z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\phi_u & 0 & 0 & 0 & 0 & \phi_0 \\ 0 & 0 & 1-\beta_x & 0 & 0 & \beta_{y_1} & \beta_{y_0} \\ 0 & 0 & 0 & 1-\sum_{i=1}^4 \mu_i & 0 & 0 & \eta_y \end{bmatrix},$$

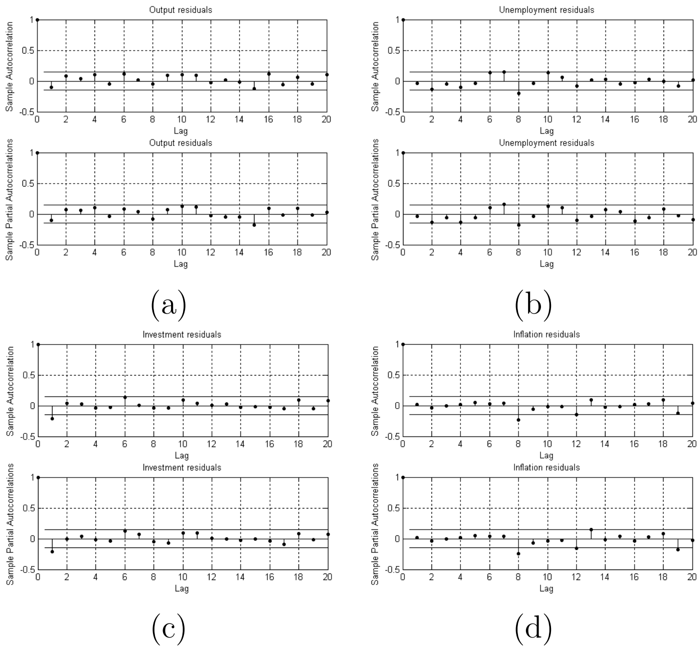
$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The parameters of model (2.6) are estimated by QML by maximizing the one-step-ahead error prediction decomposition of the Gaussian log-likelihood where the innovations and their covariances matrices are obtained by running the Kalman filter. The asymptotic distribution of the QML estimator can be found in, for example, Harvey (1989). After estimating the parameters, the Kalman filter is run again to obtain estimates of the underlying components and their corresponding PMSEs.

In this chapter, we estimate model (2.6) using the same data as in Doménech and Gómez (2006). It consists on quarterly observations of the $\log(GDP)$, the inflation rate, defined as the average inflation over the last four months, the unemployment rate which is defined as the average of the unemployment rate over the last four months and, finally, the nominal investment rate. The period analyzed is from 1948:Q1 to 2003:Q1. Preliminary analysis of the data confirms the specification proposed by Doménech and Gómez (2006). Table 2.3 reports the QML estimates of the parameters which are also very close to those reported by Doménech and Gómez (2006). Note that in the output column in Table 2.3, the estimated break in the output volatility is highly significant. There is a decrease in the volatility after 1983:Q1. Moreover, the volatility in inflation clearly has two significant breaks: a substantial increase in 1972:Q1 and, a decrease in 1983:Q1. Finally, the output gap is significant in both the investment equations and the Phillips curve. In both cases, the sign of the coefficients is positive as expected.

Figure 2.7 plots the sample autocorrelations and the partial-autocorrelations of the four one-step-ahead components of the innovations vector. They seem to have no pattern, indicating no evidence of residual autocorrelation. On the other hand, Figure 2.8 shows estimated kernel densities for the innovations and, apparently, not all residuals seem to be normally distributed. In particular, the unemployment and investment residuals seem to have skewed distributions. Therefore, it could be expected that, in this case, the parametric PMSEs based on the Gaussian assumption and the non-parametric PMSEs may differ.

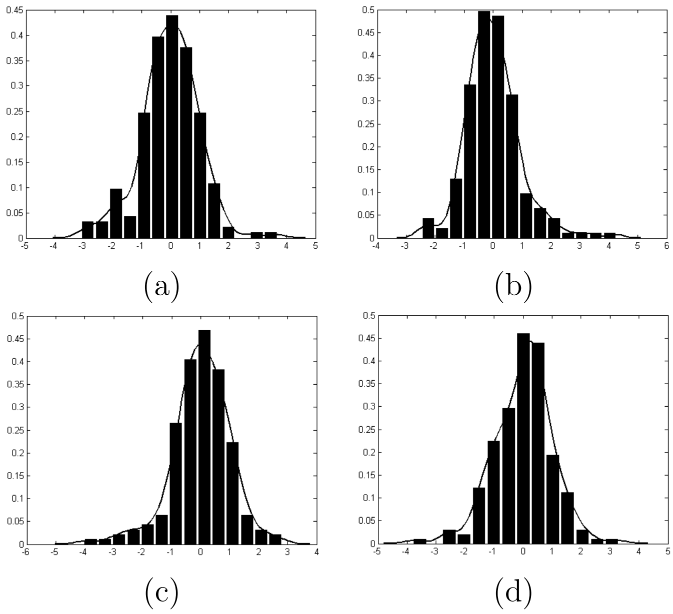
Using the estimated parameters we obtain the one-step-ahead estimates of the under-



(a) (b)

(c) (d)

Figure 2.7: Sample Autocorrelations and Partial-Autocorrelations of standardized one-step-ahead errors of the (a) output, (b) unemployment, (c) investment and (d) inflation.



(a) (b)

(c) (d)

Figure 2.8: Histogram and estimated kernel density of the standardized one-step ahead errors of the (a) output, (b) unemployment, (c) investment and (d) inflation.

Table 2.3: Parameter estimates of model (2.6).

Output		NAIRU		Investment		Inflation			
θ_1	0.7710	ϕ_0	-0.3107	β_{y0}	0.6458	η_y	0.3123	$\sigma_{\upsilon\pi}$	0.0044
	(15.73)		(-14.31)		(16.24)		(3.52)		(2.94)
θ_2	0.2359	ϕ_u	0.3653	β_{y1}	-0.6102	μ_1	0.1427	$\sigma_{\omega\pi 1}$	0.0096
	(3.02)		(6.27)		(-12.89)		(2.24)		(8.69)
$\sigma_{\omega z1}$	0.0075	$\sigma_{\upsilon u}$	0.0009	β_x	0.8253	μ_2	-0.1332	$\sigma_{\omega\pi 2}$	0.0163
	(10.81)		(3.86)		(15.05)		(-2.67)		(4.38)
$\sigma_{\omega z2}$	0.0069	$\sigma_{\omega u}$	0.0022	$\sigma_{\upsilon u}$	0.0033	μ_3	0.2413	$\sigma_{\omega\pi 3}$	0.0047
	(11.59)		(6.41)		(12.17)		(3.13)		(3.41)
$\sigma_{\omega y}$	0.0048			$\sigma_{\omega u}$	0.0023	μ_4	-0.1679		
	(14.07)				(3.17)		(-2.35)		

Note: Estimation were carried out with 221 quarterly observations from 1948:I to 2003:I. In parenthesis is the t-statistic.

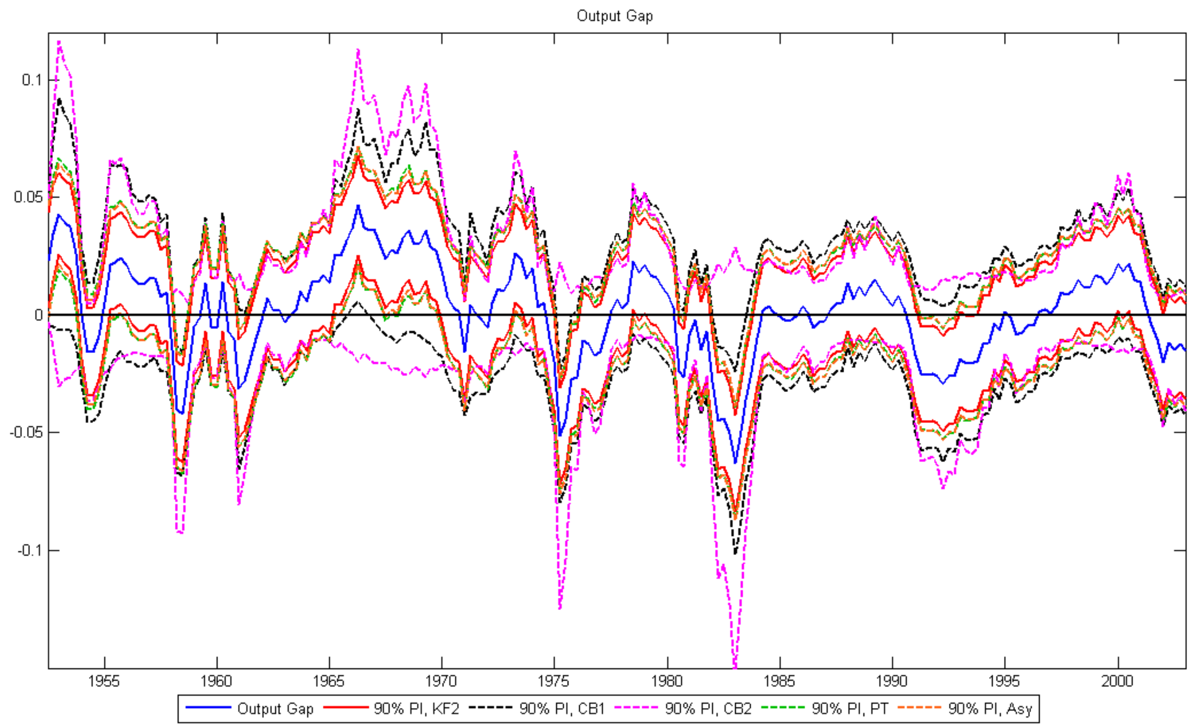


Figure 2.9: Estimated and 90% prediction intervals for the output gap.

lying components and their PMSE by running the Kalman filter[2]. The estimates of

[2]Alternatively, Doménech and Gómez (2006) implement a smoothing algorithm to estimate the unobserved components together with their PMSEs. However, they report very large correlations between smoothed and one-step-ahead estimates of the underlying components. Therefore, their estimates are comparable with those obtained in this chapter.

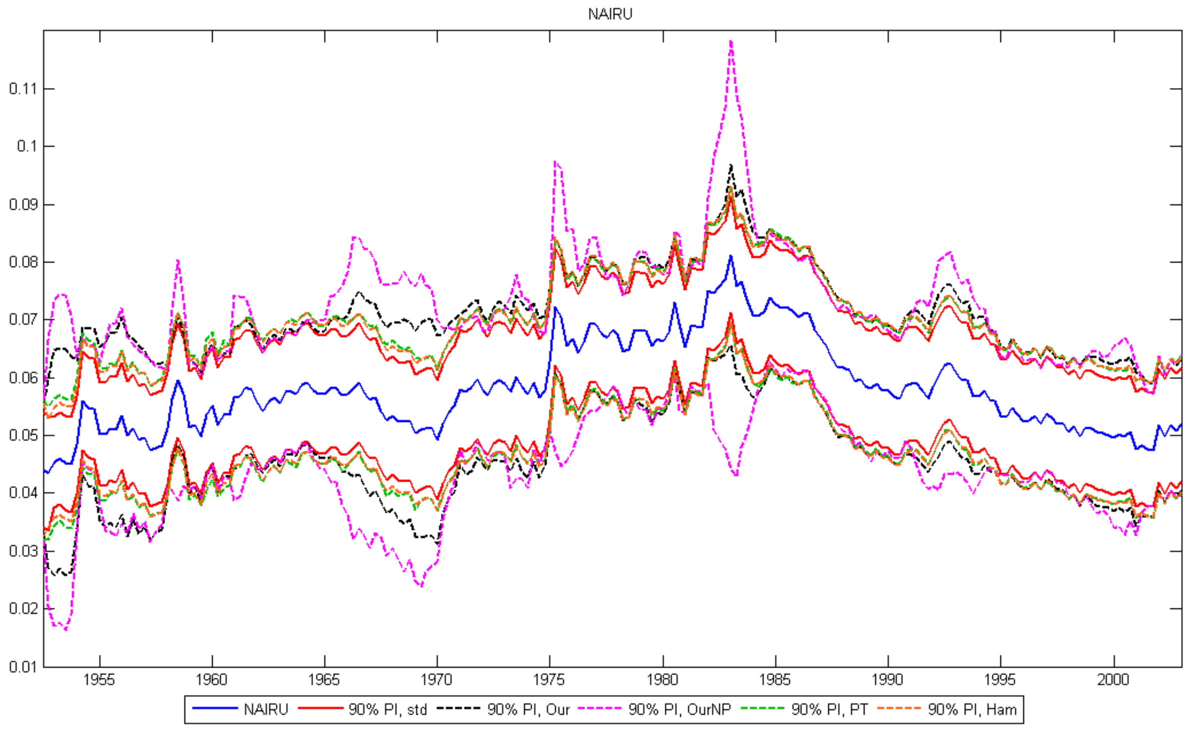


Figure 2.10: Estimated and 90% prediction intervals for the NAIRU.

the output gap, NAIRU, investment trend and the core inflation have been plotted in Figures 2.9 to 2.12 together with the 90% intervals obtained by assuming a Normal distribution as follows $\widehat{a}_{t|t-1} \pm 1.64\widehat{P}_{t|t-1}$. We also estimate the PMSE by using the asymptotic approximation proposed by Hamilton (1986), the parametric bootstrap procedure of Pfeffermann and Tiller (2005) and the two new bootstrap procedures proposed in this chapter. Table 2.4 reports the averages of the PMSEs estimated for each of the underlying components by each of the five procedures. The PMSEs obtained by the Kalman filter run with the estimated parameters and the procedure proposed by Hamilton (1986) are very similar for the NAIRU, Investment and long-run inflation. However, there is a large difference in the PMSE for output gap which is 0.0143 when estimated by the Kalman filter with estimated parameters while it is 0.0214 when incorporating the parameter uncertainty using the asymptotic distribution of the QML estimator. Furthermore, the PMSEs estimated using the bootstrap procedure proposed by Pfeffermann and Tiller (2005) assuming Gaussian errors are very similar to those obtained by using the asymptotic procedure for all variables but the investment. In this case, it is larger using the bootstrap procedure, 0.0093, while it is estimated as 0.0062 using the asymptotic approximation. Finally, the PMSEs obtained using the two bootstrap procedures proposed in this chapter are significantly larger with respect to all the alternative procedures for all four unobserved variables. Although the parametric bootstrap is based on the assumption of Gaussian errors, which seems to be not satisfied in all equations, the PMSEs estimated using the parametric and non-parametric bootstrap procedures are very similar for all variables. Only in the case of the NAIRU the PMSE estimated using the parametric bootstrap is 0.0063 while it is 0.0089 when the non-parametric procedure is implemented. Also note that, for the investment trend, the bootstrap PMSE is around five times the PMSE computed using the Kalman filter with estimated parameters. The smallest difference between the bootstrap and Kalman filter PMSE is about 30% for the core inflation. Consequently, the 90% prediction intervals based on the PMSEs proposed by Hamilton (1986) and Pfeffermann and Tiller (2005) will be wider than those based on the PMSEs of the Kalman filter with estimated parameters. Furthermore, when the bootstrap PMSEs proposed in this chapter are used for constructing prediction intervals, the resulting intervals will be still wider than for the previous procedures. Figures 2.9 to 2.12, that plot the 90% prediction intervals for the PMSEs computed by all procedures considered in this chapter, show that for the unobserved variables considered the intervals are very similar when obtained using the Kalman filter, asymptotic and the parametric bootstrap PMSE as proposed by Pfeffermann and Tiller (2005). However, there are particular moments of time when the prediction intervals of the output gap and NAIRU are clearly wider when the PMSE are computed using the procedures proposed in this thesis. Furthermore, while the differences in the prediction intervals of the unobserved core inflation are very small, the intervals corresponding to investment are much wider.

Therefore, we conclude that incorporating the parameter uncertainty may have empirical implications. Consider, for example, the usefulness of the difference between the NAIRU and the unemployment rate to identify expansions and recessions. Figure 2.13 plots the unemployment rate together with the 90% intervals of the NAIRU constructed with the Kalman filter and with the non-parametric bootstrap proposed in this chapter. The unemployment is clearly out of the Kalman filter intervals in several moments along the sample period which allows Doménech and Gómez (2006) to conclude that the NAIRU is an useful indicator of expansions and recessions. However, when the parameter uncertainty is computed taking into account the parameter uncertainty as proposed in this thesis, the unemployment is inside the prediction intervals of the NAIRU supporting the conclusion of Staiger et al. (2001) that about doubt the ability of the difference between the unemployment and the NAIRU as an indicator of expectations and recessions useful for economy policy.

2.5 Conclusions

In this chapter, we propose two new bootstrap procedures to obtain the PMSE of the Kalman filter estimator of the unobserved state in states space models which take

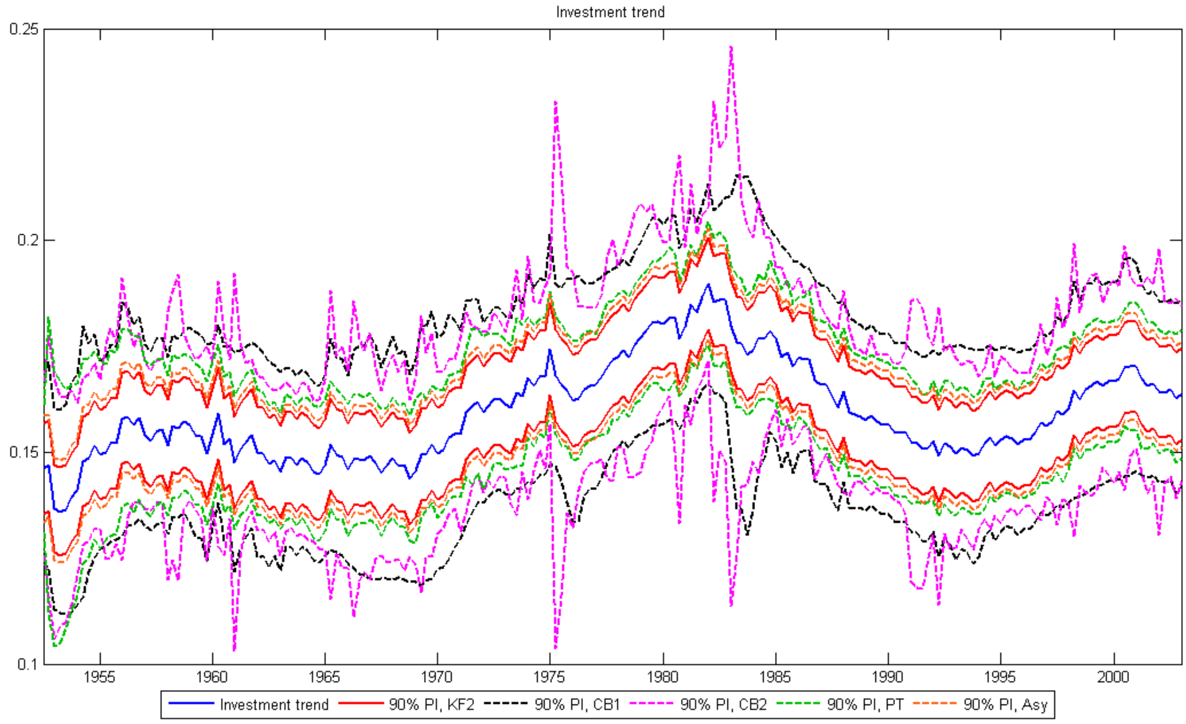


Figure 2.11: Estimated and 90% prediction intervals for the Investment trend.

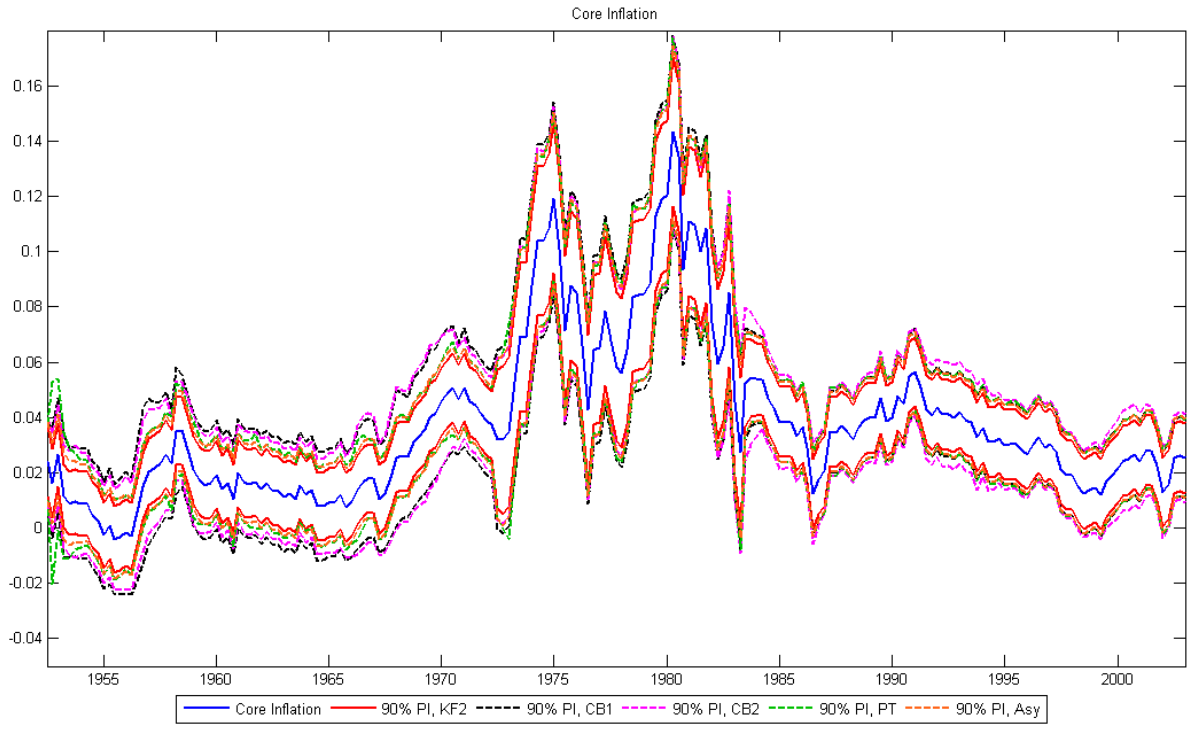


Figure 2.12: Estimated and 90% prediction intervals for the Core Inflation.

Table 2.4: Averages and standard deviations (in brackets) through time of PMSEs computed using the Kalman filter with estimated parameters (KF2), the asymptotic approximation of Hamilton (1986) (Asymptotic), the bootstrap procedure of Pfeffermann and Tiller (2005) (PT), and the parametric (CB1) and non-parametric (CB2) bootstrap procedures.

	KF2	Asymptotic	PT	CB1	CB2
Output Gap	0.0143	0.0214	0.0217	0.0238	0.0283
	[0.0012]	[0.0013]	[0.0016]	[0.0078]	[0.0095]
NAIRU	0.0050	0.0051	0.0051	0.0063	0.0089
	[0.0004]	[0.0004]	[0.0004]	[0.0021]	[0.0074]
Investment	0.0059	0.0062	0.0093	0.0212	0.0223
	[0.0019]	[0.0021]	[0.0048]	[0.0066]	[0.0097]
Inflation	0.0137	0.0140	0.0142	0.0179	0.0173
	[0.0116]	[0.0119]	[0.0121]	[0.0132]	[0.0110]

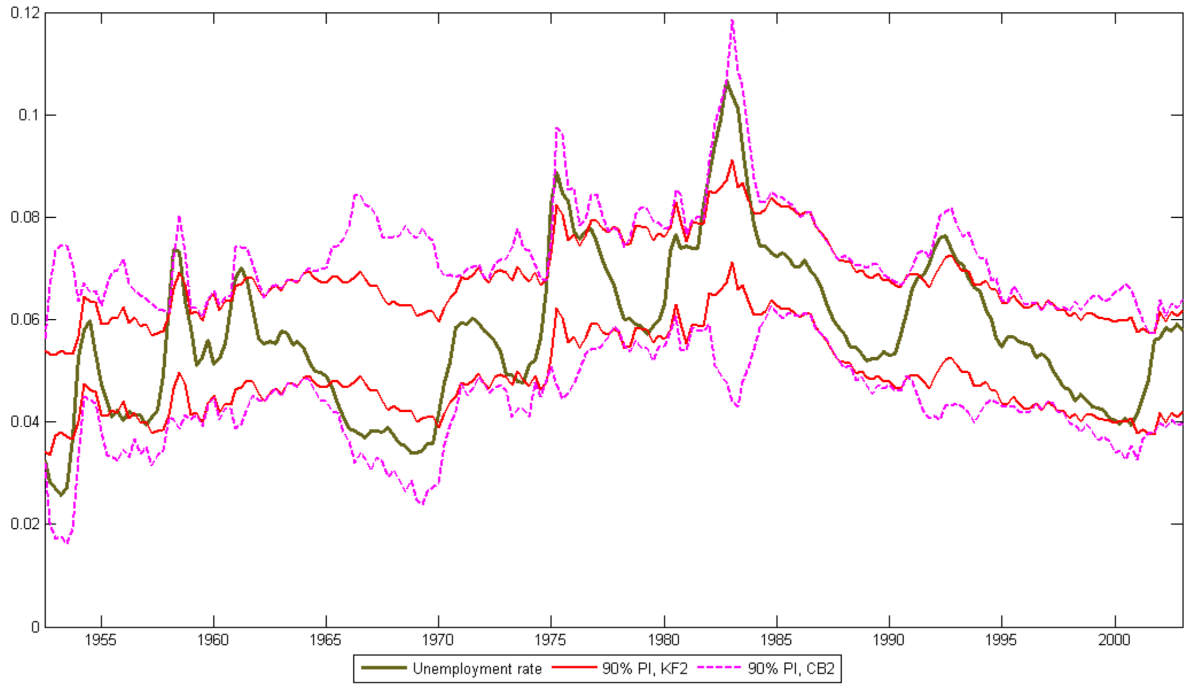


Figure 2.13: Estimated of the NAIRU, the unemployment rate and 90% prediction intervals.

into account the uncertainty attributable to parameter estimation. They have the advantage of being as simple as the procedures based on the asymptotic distribution of the parameters and, at the same time, have the good performance of alternative bootstrap procedures even in small sample sizes.

We show that our bootstrap procedures for estimating PMSE of the one-step-ahead estimator of unobserved state for time-invariant and time-variant models have better small sample properties than alternative bootstrap procedures previously proposed in the literature. The two new bootstrap PMSEs are also more accurate than the asymptotic procedures and than those obtained from the Kalman filter with estimated parameters.

It is important to note that we also show that our bootstrap procedures for estimating PMSE of the one-step-ahead estimator of the underlying components perform very well when the conditional Normality assumption is not satisfied.

Finally, our proposed bootstrap procedures are implemented to estimate the PMSE for the one-step-ahead estimator of the Output Gap, NAIRU, the trend investment rate and the core inflation for the US economy. In this case, the PMSE estimated by the new bootstrap procedures are larger than those obtained with the alternative procedures and, consequently, the prediction intervals are wider. We show that these differences may have consequences for policy makers. In particular, we put some doubts on the usefulness of the difference between the unemployment rate and the NAIRU for predicting the expansions and the recessions of the economy.

Chapter 3

Bootstrap Prediction Intervals in State Space Models: A new proposal

3.1 Introduction

In this chapter we deal with the contribution of prediction intervals of future observations that incorporate the parameter uncertainty. As mentioned in Chapter 1, Wall and Stoffer (2002) propose to use bootstrap procedures with this goal. However, their procedure requires the use of the backward representation of the model. Furthermore, its implementation is complicated by the fact that the bootstrap density of the prediction errors is obtained in two steps. First, they obtain the density that takes into account the parameter estimation uncertainty and then the density that takes into in account the variability of future innovations. Finally, these two densities are combined in the overall density of the prediction errors that is itself used to obtain the density of future observations. In this chapter we propose a bootstrap procedure to obtain directly prediction densities and, consequently, prediction intervals of future observations in state space models that incorporate the parameter uncertainty and that does not rely on the Gaussianity assumption. Furthermore, the proposed procedure does not require the backward representation. As in Wall and Stoffer (2002), our proposed bootstrap procedure is based on the innovation form of state space models. We show that the new procedure has the advantage of being much simpler than the Wall and Stoffer (2002), (WS), procedure without loosing the good behavior of bootstrap procedures. The finite sample behavior of the new intervals is compared with intervals based on the standard Kalman filter and on the WS procedure in the context of Gaussian and non-Gaussian linear state space models.

The rest of the chapter is organized as follows. Section 2 describes the new bootstrap procedure proposed for the construction of bootstrap prediction intervals in state space models. Section 3 analyzes the finite sample properties of the new procedure by means of Monte Carlo experiments. They are then compared with those of the standard and WS prediction intervals. Section 4 presents an application of the new bootstrap procedure to a real time series of the US standardized quarterly mortgages change in home equity debt outstanding, unscheduled payments. Section 5 concludes the paper with our conclusions and some suggestions for future research.

3.2 Bootstrap Prediction Intervals in State Space Models

In this section we describe the new bootstrap procedure for constructing prediction densities of the future values of the series of interest. Consider the state space model in (1.1) with constant system matrices, our proposal is to construct bootstrap prediction intervals approximating the conditional distribution of Y_{T+k} by the distribution of bootstrap replicates that incorporate simultaneously the variability due to parameter estimation and the uncertainty due to unknown future innovations without using the backward filter.

The proposed procedure consists on the following steps:

Step 1: Estimate the parameters of model (1.1) by QML, $\widehat{\theta}$, and obtain the standardized innovations $\left\{\widehat{V}_t^s; 1 \leq t \leq T\right\}$.

Step 2: Construct a sequence of bootstrap standardized innovations $\left\{\widehat{V}_t^{s*}; 1 \leq t \leq T+K\right\}$ via random draws with replacement from the standardized innovations, $\widehat{V}_t^s$.

Step 3: Compute a bootstrap replicate $\left\{\widehat{Y}_t^*; 1 \leq t \leq T\right\}$ by means of the IF using $\widehat{V}_t^{s*}$ and the estimated parameters, $\widehat{\theta}$ as follows

$$\begin{aligned} \widehat{a}_{t+1|t}^* &= \widehat{T}\widehat{a}_{t|t-1}^* + \widehat{c} + \widehat{T}\widehat{P}\widehat{Z}'\widehat{F}^{-1}V_t^*, \\ \widehat{Y}_t^* &= \widehat{Z}_t a_{t|t-1}^* + \widehat{d} + V_t^*, \end{aligned}$$

with $\widehat{a}_0^* = \widehat{a}_1$. Estimate the corresponding bootstrap parameters, $\widehat{\theta}^*$, and run the Kalman filter with $\widehat{\theta}^*$ in order to obtain bootstrap replicates of the state vector at time T which incorporate the uncertainty due to parameter estimation, $\widehat{a}_{T|T-1}^*$.

Step 4: Obtain the conditional bootstrap predictions $\left\{\widehat{Y}_{T+k|T}^*; 1 \leq k \leq K\right\}$ by the following expressions

$$\begin{aligned} \widehat{a}_{T+k|T}^* &= \widehat{T}^{*k}\widehat{a}_{T|T-1}^* + k\widehat{c}^* + \sum_{j=0}^{k-1}\widehat{T}^{*k-1-j}\widehat{P}^*\widehat{Z}^{*\prime}\widehat{F}^{*-1}\widehat{V}_{T+j}^*, \\ \widehat{Y}_{T+k|T}^* &= \widehat{Z}^*\widehat{T}^{*k}\widehat{a}_{T|T-1}^* + \widehat{Z}^*k\widehat{c}^* + \widehat{d}^* \\ &+ \widehat{Z}^*\sum_{j=0}^{k-1}\widehat{T}^{*k-1-j}\widehat{P}^*\widehat{Z}^{*\prime}\widehat{F}^{*-1}\widehat{V}_{T+j}^* + \widehat{V}_{T+k}^*, k = 1, \ldots, \end{aligned}$$

where $\widehat{V}_T^* = Y_T - \widehat{Z}^*\widehat{a}_{T|T-1}^* - \widehat{Z}^*k\widehat{c}^*$.

Steps 2 to 4 are repeated B times.

The empirical distribution of $\widehat{Y}^*_{T+k|T}$ incorporates both the variability due to unknown future innovations and the variability due to parameter estimation in just one Step. The procedure above, denoted as state space Bootstrap (SSB), has three advantages over the WS procedure. First, it does not require to use the backward representation. Second, it is simpler as a unique set of bootstrap replicates of future observations is required instead of two as in the WS procedure. Third, unlike the WS procedure, in Step 5, we do not fix $\widehat{a}^*_{T|T-1} = \widehat{a}_{T|T-1}$ because this value depends on the estimated parameters, and therefore it should be allowed to vary among bootstrap replicates in order to incorporate the uncertainty due to parameter estimation.

Finally, bootstrap prediction intervals are constructed directly by the percentile method[1]. Hence, bootstrap prediction intervals are given by

$$\left[Q^*_{\alpha/2,\widehat{Y}^*_{T+k|T}},\ Q^*_{1-\alpha/2,\widehat{Y}^*_{T+k|T}}\right] \tag{3.1}$$

where $Q^*_{\alpha/2,\widehat{Y}^*_{T+k|T}}$ is the $\frac{\alpha}{2}$-percentile of the empirical bootstrap distribution of the k-step ahead prediction of Y_{T+k}.

3.3 Finite sample properties

In this section, we analyze the finite sample properties of the SSB prediction intervals and compare them with those of the ST and WS intervals[2]. We consider three different Monte Carlo designs based on the RWN model in (2.4) with different assumptions about the distribution of the disturbance associated with the measurement equation ε_t. In particular, we consider a Gaussian white noise with unit variance, a centered and re-

[1]We try alternative methods as the bias-corrected and the acceleration bias-corrected with similar results; see Efron (1987) for a definition of these intervals.

[2]All programs for maximizing the log-likelihood and subsequent estimation of the unobserved components and PMSEs were written in MATLAB.

scaled Chi-square with 1 degree of freedom and a Student-t distribution with 5 degrees of freedom. For the three cases, simulation results are based on $R = 1000$ replicates of series of sizes $T = 50, 100$ and 500. The parameters of the model have been chosen to cover a wide range of different situations from cases in which the noise is large relative to the signal, i.e. q is small, to cases in which q is large. In particular, we consider $q = \{0.1, 1, 2\}$. For each simulated series, $\{y_1^r, \ldots, y_T^r\}$, $r = 1, 2, \ldots, R$, we first generate $B = 1000$ observations of y_{T+k}^r for prediction horizons $k = 1, 5$ and 15, and then obtain, 95% prediction intervals computed using, the ST intervals in (1.12), the WS intervals in (1.17) and the SSB intervals in (3.1). Finally, we compute the coverage of each of these intervals as well as the length and the percentage of observations left out on the right size and on the left size of the limits of the prediction intervals.

Table 3.1 reports the Monte Carlo averages of these quantities when both disturbances are Gaussian, and the predictions are calculated for $k = 1, 5$ and 15 prediction horizons. The table shows that, as expected given the Gaussianity of the model, the three procedures are very similar. However, The SSB procedure seems to be slightly better specially when the sample size is small and the prediction horizon increases. This improvement in the coverage seems to be due to that our procedure is incorporating the parameter uncertainty. This result is illustrated in Figures 3.1 to 3.3 that plot kernel estimates of the ST, WS and SSB densities for the 1, 5 and 15-steps ahead predictions for one particular series generated by each of the three models considered with $T = 50, 100$ and 500 together with the empirical density. Note that when the signal to noise ratio is small, i.e. $q = 0.1$, the density of the SSB procedure seems to be more similar to the

Table 3.1: Monte Carlo Average coverages, length and percentage of observations left out on the right and on the left of the 95% prediction intervals constructed using ST, WS and SSB when ε_t is $\mathcal{N}(0,1)$, η_t is $\mathcal{N}(0,q)$.

	k	Mean coverage			Mean coverage in tails			Mean length		
		ST	WS	SSB	ST Below/Above	WS Below/Above	SSB Below/Above	ST	WS	SSB
$T=50$										
$q=0.1$	1	0.927	0.935	0.936	0.036/0.037	0.030/0.035	0.031/0.033	4.530	4.597	4.774
	5	0.927	0.940	0.943	0.036/0.037	0.029/0.031	0.028/0.029	5.182	5.285	5.539
	15	0.915	0.928	0.940	0.042/0.042	0.035/0.037	0.030/0.031	6.460	6.633	7.052
$q=1$	1	0.936	0.923	0.928	0.029/0.035	0.036/0.041	0.036/0.035	6.157	6.250	6.280
	5	0.927	0.921	0.938	0.035/0.039	0.037/0.042	0.032/0.031	9.722	9.718	10.274
	15	0.914	0.909	0.934	0.041/0.045	0.043/0.047	0.033/0.033	15.258	15.194	16.469
$q=2$	1	0.938	0.930	0.930	0.032/0.029	0.036/0.034	0.036/0.034	7.424	7.56	7.433
	5	0.926	0.924	0.931	0.037/0.036	0.038/0.038	0.034/0.034	12.849	12.880	13.088
	15	0.918	0.915	0.930	0.041/0.041	0.042/0.042	0.035/0.035	20.889	20.830	21.632
$T=100$										
$q=0.1$	1	0.945	0.941	0.943	0.025/0.030	0.031/0.028	0.026/0.031	4.569	4.576	4.618
	5	0.945	0.942	0.948	0.025/0.030	0.030/0.028	0.024/0.029	5.206	5.238	5.334
	15	0.938	0.938	0.945	0.029/0.033	0.032/0.030	0.026/0.030	6.498	6.575	6.743
$q=1$	1	0.944	0.940	0.939	0.028/0.028	0.030/0.029	0.030/0.031	6.271	6.314	6.278
	5	0.939	0.937	0.942	0.031/0.030	0.032/0.031	0.029/0.029	9.874	9.873	10.120
	15	0.934	0.932	0.940	0.033/0.033	0.034/0.034	0.030/0.030	15.547	15.521	16.165
$q=2$	1	0.945	0.937	0.939	0.028/0.027	0.032/0.030	0.031/0.030	7.476	7.537	7.460
	5	0.939	0.938	0.939	0.030/0.030	0.031/0.031	0.031/0.031	13.137	13.155	13.210
	15	0.935	0.935	0.937	0.032/0.032	0.032/0.033	0.031/0.031	21.509	21.539	21.758
$T=500$										
$q=0.1$	1	0.946	0.948	0.945	0.027/0.027	0.025/0.028	0.028/0.027	4.592	4.577	4.582
	5	0.946	0.947	0.946	0.026/0.028	0.025/0.028	0.027/0.027	5.217	5.206	5.223
	15	0.946	0.945	0.945	0.026/0.029	0.026/0.029	0.027/0.028	6.515	6.477	6.511
$q=1$	1	0.948	0.948	0.947	0.029/0.023	0.027/0.025	0.027/0.025	6.339	6.335	6.314
	5	0.948	0.947	0.947	0.027/0.025	0.027/0.026	0.028/0.025	10.075	10.049	10.073
	15	0.947	0.946	0.947	0.027/0.026	0.027/0.027	0.027/0.026	15.956	15.919	15.944
$q=2$	1	0.947	0.945	0.947	0.027/0.026	0.029/0.026	0.027/0.026	7.563	7.546	7.540
	5	0.948	0.948	0.948	0.027/0.027	0.027/0.025	0.027/0.027	13.418	13.446	13.387
	15	0.947	0.948	0.947	0.027/0.026	0.026/0.026	0.026/0.027	22.066	22.112	22.051

Table 3.2: Monte Carlo Average coverages, length and percentage of observations left out on the right and on the left of the 95% prediction intervals constructed using ST, WS and SSB when ε_t is $\chi^2_{(1)}$, η_t is $\mathcal{N}(0, q)$.

Case	k	Mean coverage ST	Mean coverage WS	Mean coverage SSB	Mean coverage in tails ST Below/Above	Mean coverage in tails WS Below/Above	Mean coverage in tails SSB Below/Above	Mean length ST	Mean length WS	Mean length SSB
$T = 50$										
$q = 0.1$	1	0.941	0.940	0.942	0.010/0.049	0.030/0.030	0.027/0.031	4.513	4.909	4.734
	5	0.943	0.934	0.946	0.013/0.044	0.039/0.027	0.027/0.026	5.221	5.507	5.596
	15	0.930	0.919	0.950	0.025/0.045	0.053/0.029	0.027/0.023	6.572	6.665	7.329
$q = 1$	1	0.935	0.932	0.934	0.026/0.039	0.034/0.034	0.034/0.032	6.200	6.514	6.459
	5	0.926	0.926	0.930	0.034/0.040	0.040/0.034	0.038/0.032	9.682	9.803	9.919
	15	0.913	0.914	0.923	0.042/0.045	0.041/0.036	0.045/0.041	15.126	15.176	15.597
$q = 2$	1	0.937	0.933	0.932	0.028/0.035	0.034/0.034	0.035/0.033	7.378	7.714	7.575
	5	0.927	0.924	0.927	0.035/0.038	0.040/0.036	0.038/0.035	12.805	12.897	12.957
	15	0.919	0.917	0.923	0.040/0.041	0.043/0.040	0.040/0.037	20.839	20.880	21.236
$T = 100$										
$q = 0.1$	1	0.947	0.939	0.943	0.006/0.048	0.033/0.028	0.027/0.029	4.552	4.773	4.710
	5	0.946	0.937	0.942	0.010/0.043	0.037/0.026	0.031/0.027	5.196	5.356	5.414
	15	0.939	0.929	0.944	0.021/0.040	0.045/0.026	0.032/0.024	6.491	6.597	6.912
$q = 1$	1	0.942	0.939	0.943	0.021/0.037	0.030/0.030	0.030/0.027	6.244	6.483	6.501
	5	0.937	0.936	0.939	0.028/0.034	0.035/0.029	0.022/0.035	9.813	9.919	10.017
	15	0.932	0.930	0.935	0.033/0.035	0.037/0.033	0.028/0.032	15.438	15.445	15.742
$q = 2$	1	0.947	0.944	0.945	0.022/0.031	0.028/0.028	0.027/0.028	7.507	7.685	7.686
	5	0.942	0.941	0.943	0.028/0.030	0.031/0.028	0.030/0.027	13.220	13.307	13.399
	15	0.938	0.937	0.940	0.030/0.031	0.033/0.030	0.032/0.028	21.659	21.752	21.941
$T = 500$										
$q = 0.1$	1	0.948	0.940	0.950	0.006/0.045	0.033/0.026	0.023/0.027	4.575	4.697	4.707
	5	0.948	0.939	0.947	0.011/0.041	0.036/0.025	0.028/0.025	5.184	5.272	5.314
	15	0.946	0.937	0.946	0.019/0.035	0.039/0.024	0.031/0.024	6.455	6.507	6.631
$q = 1$	1	0.947	0.947	0.948	0.020/0.033	0.020/0.033	0.027/0.026	6.338	6.492	6.472
	5	0.948	0.948	0.948	0.024/0.028	0.029/0.023	0.027/0.025	10.073	10.181	10.137
	15	0.947	0.946	0.947	0.025/0.027	0.029/0.024	0.028/0.025	15.952	15.983	15.957
$q = 2$	1	0.944	0.945	0.944	0.026/0.030	0.027/0.028	0.029/0.026	7.554	7.648	7.636
	5	0.947	0.947	0.948	0.026/0.027	0.028/0.025	0.028/0.024	13.369	13.447	13.466
	15	0.947	0.947	0.948	0.026/0.027	0.028/0.025	0.027/0.025	21.968	21.992	22.085

Table 3.3: Monte Carlo Average coverages, length and percentage of observations left out on the right and on the left of the 95% prediction intervals constructed using ST, WS and SSB when ε_t is Student-t with 5 degree of freedom, η_t is $\mathcal{N}(0, q)$.

Case	k	Mean coverage ST	Mean coverage WS	Mean coverage SSB	Mean coverage in tails ST Below/Above	Mean coverage in tails WS Below/Above	Mean coverage in tails SSB Below/Above	Mean length ST	Mean length WS	Mean length SSB
$T = 50$										
$q = 0.1$	1	0.942	0.939	0.945	0.027/0.031	0.030/0.031	0.026/0.029	4.522	4.499	4.799
	5	0.941	0.938	0.946	0.029/0.029	0.031/0.031	0.028/0.026	5.195	5.013	5.629
	15	0.929	0.931	0.951	0.029/0.045	0.033/0.034	0.027/0.022	6.491	6.951	7.351
$q = 1$	1	0.933	0.930	0.938	0.028/0.038	0.037/0.033	0.028/0.034	6.216	6.641	6.531
	5	0.930	0.927	0.932	0.032/0.038	0.040/0.033	0.034/0.034	9.741	9.900	9.971
	15	0.920	0.917	0.926	0.038/0.042	0.044/0.039	0.037/0.036	15.252	15.269	15.651
$q = 2$	1	0.931	0.929	0.935	0.033/0.036	0.038/0.033	0.033/0.032	7.402	7.851	7.727
	5	0.930	0.926	0.934	0.034/0.037	0.039/0.035	0.033/0.033	12.902	13.048	13.004
	15	0.921	0.919	0.926	0.039/0.040	0.042/0.039	0.037/0.037	21.026	21.010	21.253
T = 100										
$q = 0.1$	1	0.946	0.940	0.946	0.023/0.031	0.032/0.028	0.026/0.028	4.548	4.758	4.780
	5	0.945	0.934	0.945	0.027/0.028	0.038/0.027	0.030/0.025	5.191	5.308	5.489
	15	0.938	0.927	0.945	0.022/0.040	0.046/0.027	0.032/0.023	6.485	6.519	6.969
$q = 1$	1	0.938	0.937	0.940	0.026/0.037	0.035/0.028	0.030/0.030	6.296	6.677	6.548
	5	0.939	0.938	0.938	0.028/0.033	0.035/0.027	0.034/0.028	9.965	10.114	10.089
	15	0.936	0.934	0.934	0.031/0.033	0.036/0.030	0.035/0.031	15.721	15.714	15.845
$q = 2$	1	0.943	0.945	0.941	0.022/0.035	0.025/0.030	0.030/0.029	7.503	7.937	7.821
	5	0.939	0.938	0.937	0.029/0.032	0.032/0.029	0.033/0.030	13.158	13.321	13.325
	15	0.935	0.934	0.935	0.032/0.033	0.034/0.032	0.034/0.031	21.529	21.585	21.728
T = 500										
$q = 0.1$	1	0.942	0.935	0.947	0.028/0.030	0.039/0.026	0.026/0.027	4.549	4.720	4.717
	5	0.942	0.932	0.943	0.029/0.030	0.042/0.026	0.032/0.025	5.169	5.255	5.329
	15	0.941	0.933	0.944	0.022/0.037	0.042/0.025	0.033/0.023	6.456	6.516	6.672
$q = 1$	1	0.942	0.945	0.947	0.022/0.036	0.027/0.027	0.027/0.026	6.324	6.591	6.570
	5	0.946	0.946	0.945	0.024/0.030	0.030/0.024	0.030/0.025	10.055	10.185	10.122
	15	0.946	0.945	0.944	0.026/0.028	0.030/0.025	0.030/0.025	15.926	15.990	15.896
$q = 2$	1	0.944	0.948	0.948	0.025/0.032	0.026/0.026	0.026/0.026	7.542	7.856	7.827
	5	0.947	0.948	0.947	0.026/0.027	0.026/0.026	0.026/0.026	13.372	13.675	13.555
	15	0.946	0.949	0.945	0.027/0.027	0.026/0.025	0.029/0.026	21.984	22.186	22.091

empirical densities than those of the other procedures. For a larger signal-to-noise ratio the kernel for the three procedures are similar each other and similar to the empirical densities. Therefore, it seems that the parameter uncertainty is more important when the signal-to-noise ratio is small, the sample size is small and the horizon is large.

Table 3.2, that reports the results when ε_t is $\chi^2_{(1)}$ and η_t is Gaussian, shows that the mean coverage of the ST intervals is close to the nominal. However, they are not able of dealing with the asymmetry in the distribution of ε_t. The average coverage in the left tail is smaller than in the right tail. The difference between the coverage in both tails is larger in the model with $q = 0.1$ where the signal is relatively small with respect to the noise which has a non-Gaussian distribution. Note that the lack of capability of the ST intervals to deal with the asymmetry in the distribution of ε_t is larger when the sample size increases. On the other hand, the coverages of the WS and SSB intervals are rather similar with SSB being slightly closer to the nominal, for almost all models and sample sizes considered. Both bootstrap intervals are able to cope with the asymmetry of the distribution of ε_t. Consequently, according to the results reported in Table 3.2, using the much simpler SSB method does not imply a worse performance of the prediction intervals. Figures 3.4 to 3.6 illustrate these results plotting the kernel density of the simulated y_{T+1}, y_{T+5} and y_{T+15} together with the ST, WS and SSB densities obtained with a particular series generated by each of the models and sample sizes considered. These figures also illustrate the lack of fit of the ST density when $q = 0.1$ and 1. On the other hand, the shapes of the WS and SSB densities are similar, with SSB being always closer to the empirical. When the horizon of prediction increases, the shape of the empirical densities are clearly non-Normal and both bootstrap procedures are still able of capturing this feature.

Finally, Table 3.3 reports the results when ε_t is Student-t and η_t is Gaussian. Note that these results are very similar to those obtained for the Gaussian case. The three procedures have similar behavior in all sample size considered. However, SSB procedure seems to have slightly better performance than the alternatives considered specially in the small sample size.

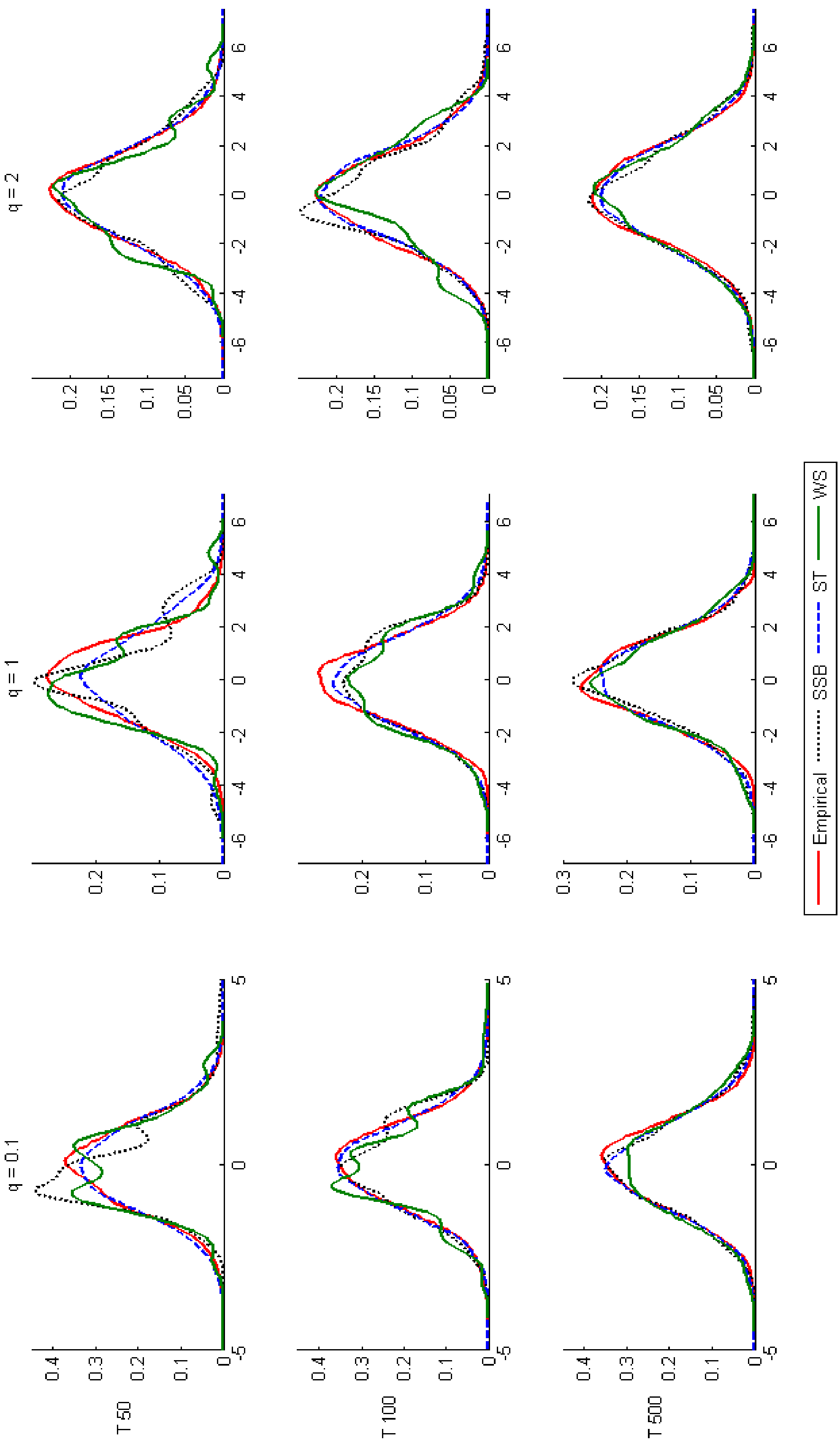


Figure 3.1: Kernel estimates densities of y_{T+k} for $k = 1$. Normal case.

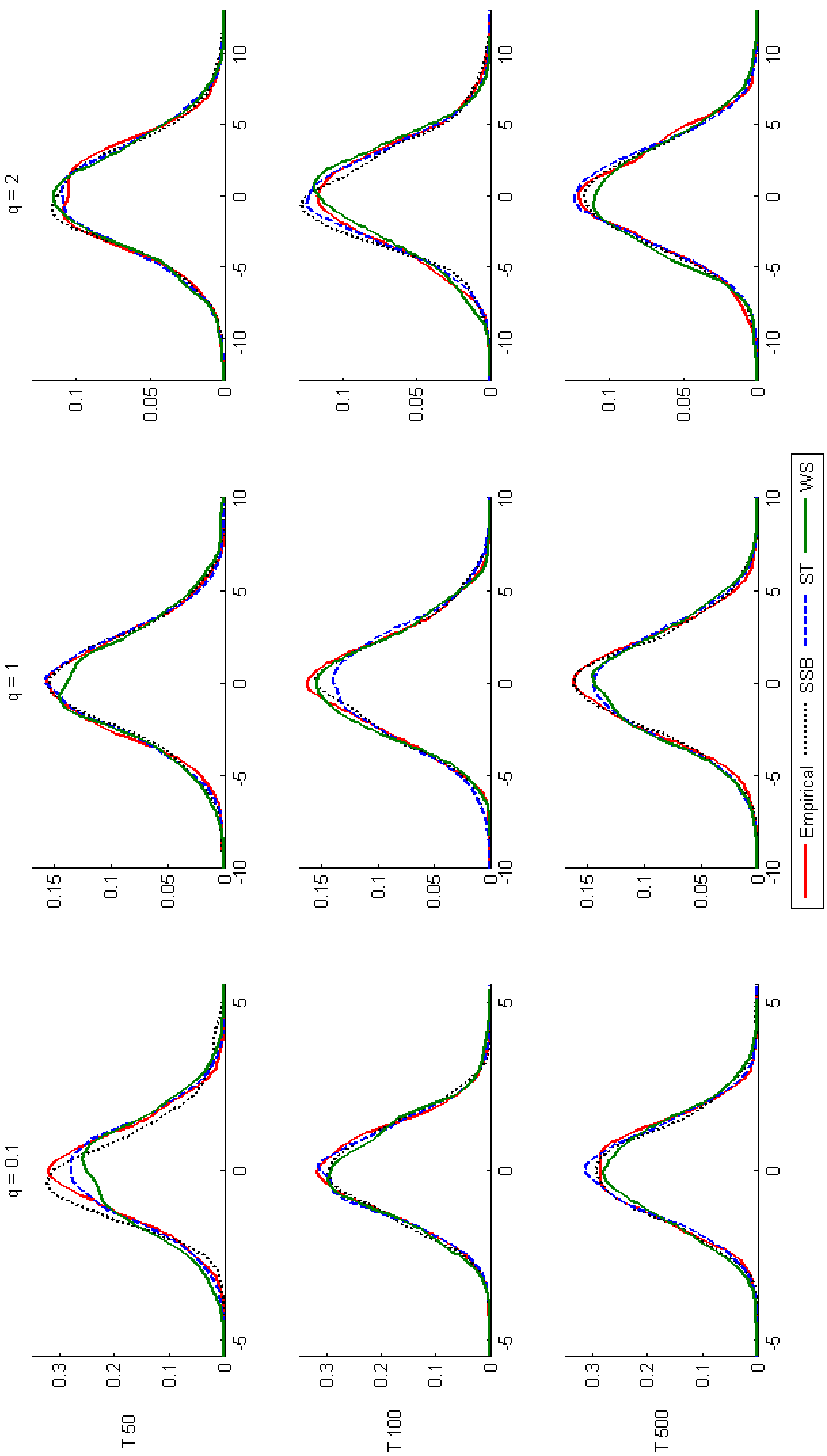


Figure 3.2: Kernel estimates densities of y_{T+k} for $k = 5$. Normal case.

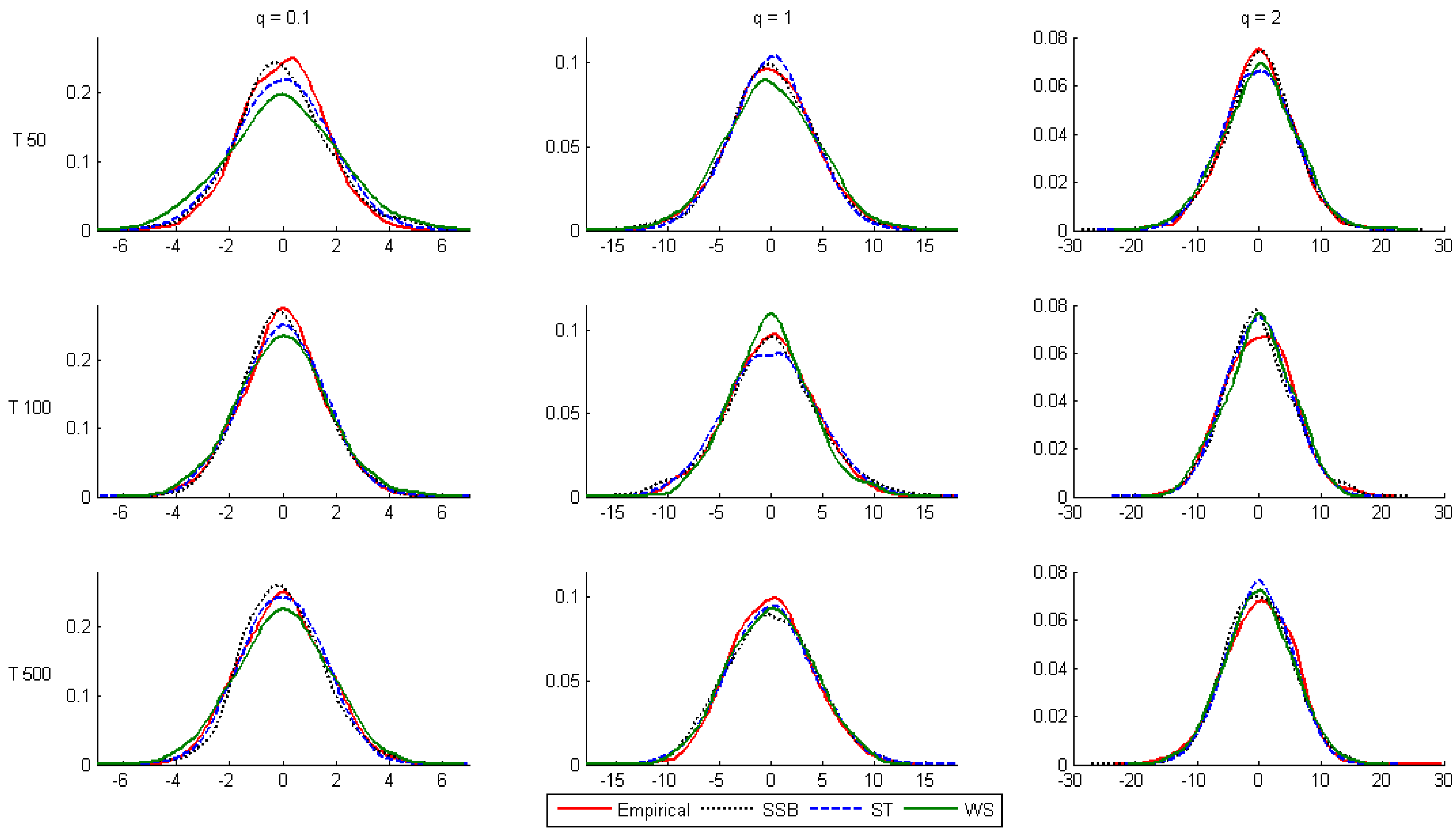


Figure 3.3: Kernel estimates densities of y_{T+k} for $k = 15$. Normal case.

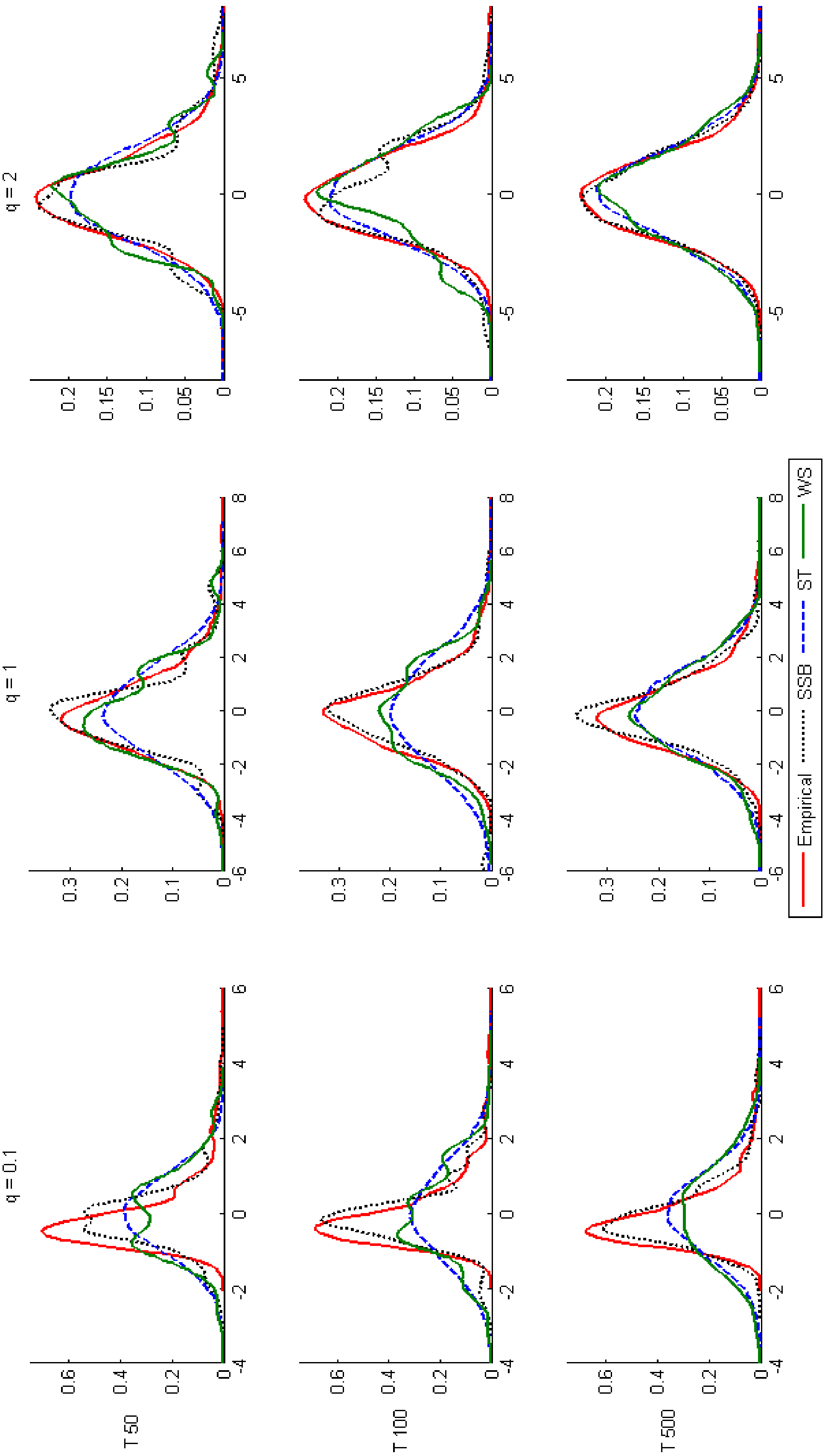


Figure 3.4: Kernel estimates densities of y_{T+k} for $k = 1$. $\chi^2_{(1)}$ case.

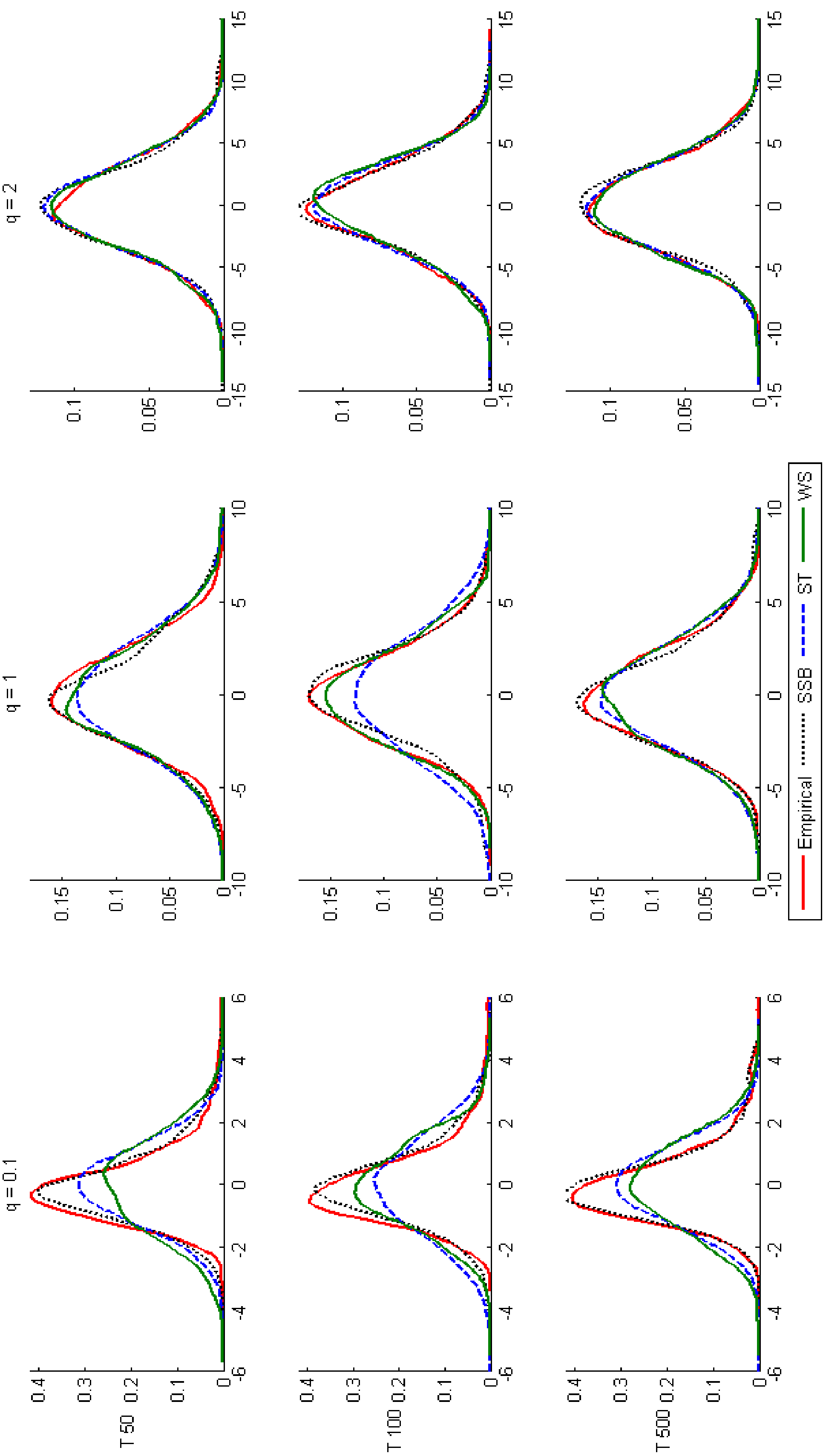


Figure 3.5: Kernel estimates densities of y_{T+k} for $k = 5$. $\chi^2_{(1)}$ case.

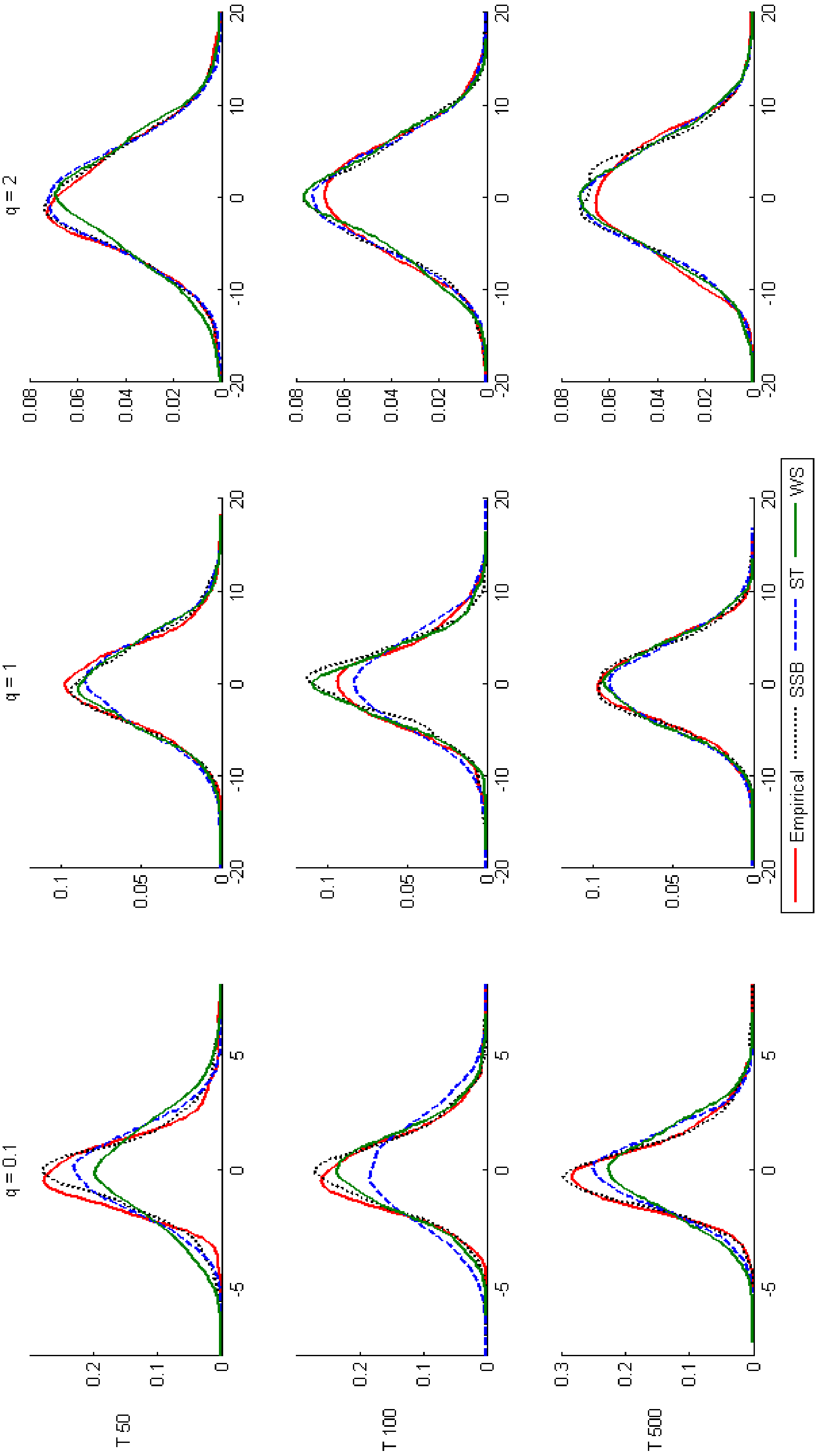


Figure 3.6: Kernel estimates densities of y_{T+k} for $k = 15$. $\chi^2_{(1)}$ case.

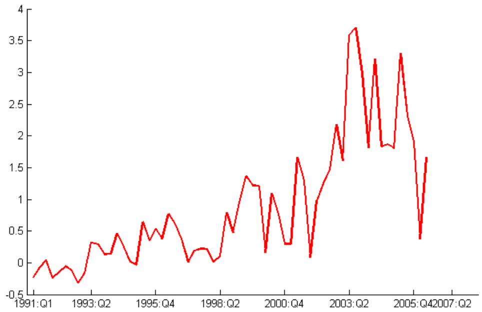
3.4 Empirical Application

We illustrate the performance of the proposed procedure to construct bootstrap prediction intervals by implementing it on the standardized quarterly mortgages change in home equity debt outstanding, unscheduled payments, observed from 1st quarter of 1991 to the 2nd quarter of 2007 (Mortgages). The series, plotted in panel (a) of Figure 3.7, is clearly not stationary. Its first differences are plotted in panel (b) together with its correlogram and partial correlogram, in panel (c). The pattern of the sample correlations and the partial correlations suggests that a moving average process of order one may represent adequately the dependence of the first differences of the series. Consequently, the RWN model in (2.4) could be adequate to represent the dynamic dependence of the series of Mortgages. Table 3.4, that reports several descriptive statistics for the first differences of Mortgages, shows that the series has excess kurtosis and positive asymmetry. The non-Gaussian distribution is reflected in the small p-values for the Jarque-Bera and the Lilliefors tests for Normality.

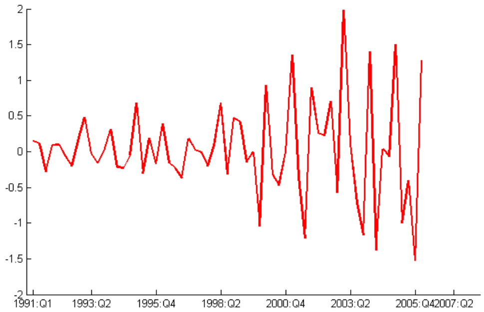
Table 3.4: Descriptive statistics

Series	Δ (Mortgage) (USD billions)
Sample Size	65
Mean	0.02
Standard Dev.	0.65
Skewness	0.38
Kurtosis	4.16
Jarque-Bera (p-value)	0.008
Lillilifors (p-value)	0.004

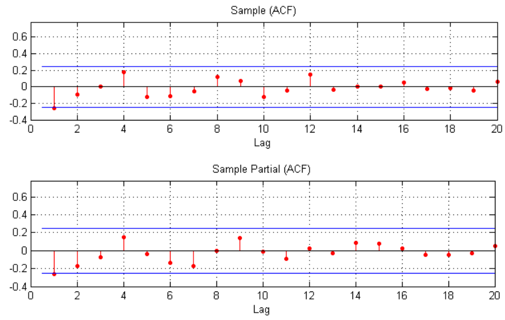
The parameters of the RWN are estimated by using the observations from the 1st quarter of 1991 up to the 1st quarter of 2001, $T = 61$, leaving the rest of them for



(a)



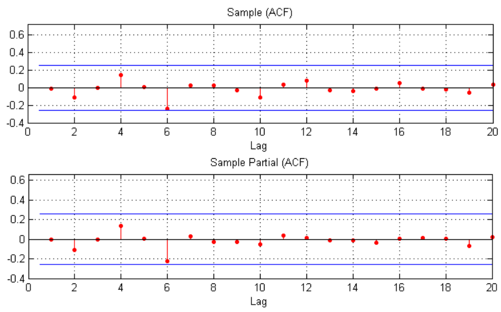
(b)



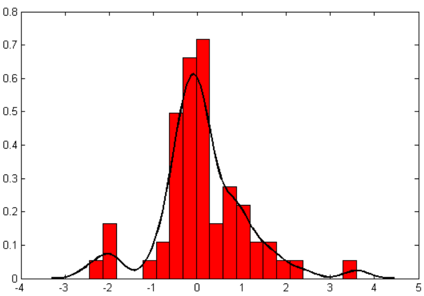
(c)

Figure 3.7: (a) The Mortgages series. (b) First difference of Mortgages. (c) Sample Autocorrelation and Partial-Autocorrelation of the first difference of the Mortgages data.

evaluating the forecast performance of the procedure.



(a)



(b)

Figure 3.8: (a) Sample Autocorrelation and Partial-Autocorrelation of standardized one-step-ahead error. (b) Empirical density and histogram for the standardized one-step-ahead error.

The QML estimates of the parameters are given by $\widehat{\sigma}_{\varepsilon}^{2} = 0.126$ and $\widehat{q} = 0.671$. These estimates are used for running the Kalman filter, to obtain estimates the innovations and their variances. Figure 3.8 plots the correlogram and partial correlogram and a kernel estimate of the density of the within sample standardized one-step ahead errors. The correlations and partial correlations are not any longer significant. However, the density of the errors suggests that they are obviously far from Normality. Therefore, the RWN model seems appropriate to represent the dependencies in the conditional mean of the Mortgages series. However, the prediction of future values should be carried out by a procedure that takes into in account the non-Normality of the errors. We construct prediction intervals up to 5 steps ahead using the ST, WS and SSB procedures. The resulting intervals are plotted in Figure 3.9 together with the observed values of the Mortgages series. First, observe that the two bootstrap procedures generate very similar intervals which are wider than the ST intervals, as expected given that they incorporate the uncertainty due to parameter estimation. For two prediction horizons, the observations corresponding to the 2nd quarter of 2006 and the 1st quarter of 2007, fall outside the ST prediction interval. However, both bootstrap procedures still contain these two values. It is important to note that although bootstrap procedures are computational intensive, in this application with $B = 2000$ bootstrap replicates, the BSS procedure requires 110 seconds using a MATLAB algorithm in an AMD Athlon 2.00GHz processor of a PC desktop with 2.00Gb of RAM. However, the Wall and Stoffer (2002) bootstrap procedure requires 160 seconds. There is a reduction of 31% in the computer time required.

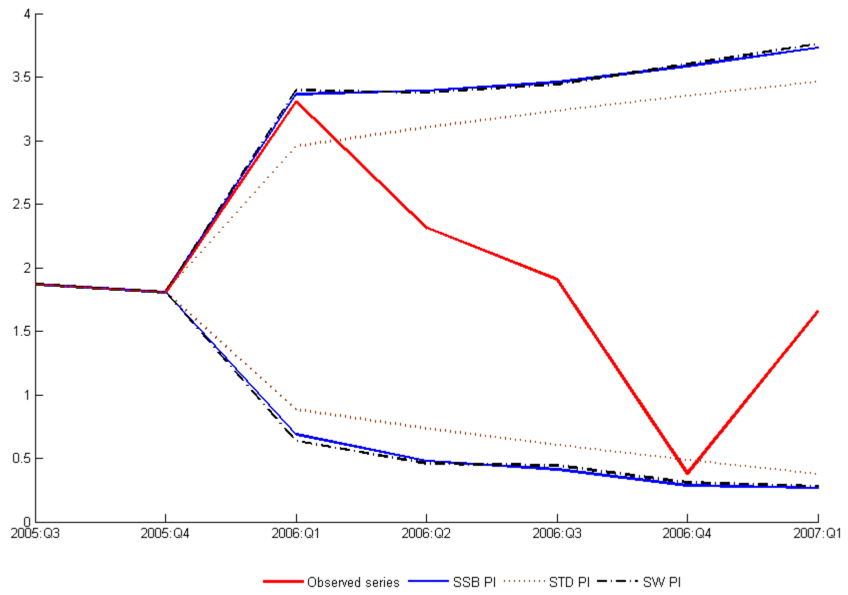


Figure 3.9: Bootstrap and standard prediction intervals for the out of sample forecasting evaluation for Mortgage series.

3.5 Conclusion

This chapter proposes a new procedure to obtain bootstrap prediction intervals in the context of state space models. Bootstrap intervals are of great interest when predicting future values of a series of interest as they are able to incorporate parameter uncertainty and do not rely on any particular assumption on the error distribution.

The procedure proposed in this chapter has three advantages over previous procedures available in the literature. First, it is based on obtaining directly the density of future observations instead of the density of the errors. Furthermore, this density is obtained in one single step and incorporates simultaneously the uncertainty due to the parameters estimation and the uncertainty due to the error distribution. Finally and more important, the bootstrap procedure proposed in this chapter does not rely on the backward representation. As a consequence, our procedure is much simpler from a computational point of view and can be extended to models without a backward representation.

We analyze the small sample behavior of the proposed bootstrap intervals and compare it with those of the intervals proposed by Wall and Stoffer (2002) and the intervals based on assuming known parameters and a Normal distribution of the errors. We show that our procedure, although much simpler, has slightly better properties than the bootstrap intervals of Wall and Stoffer (2002). As expected, we also show that bootstrap intervals are more adequate than standard intervals mainly in the presence of non-Normal errors. In general, the standard intervals are thinner than expected to have the nominal coverage and cannot deal with asymmetries.

Finally, our proposed bootstrap procedure to obtain prediction intervals in state space models is illustrated by implementing it to obtain intervals for future values of a series of Mortgages modelled by the RWN model. We show that there is an important improvement in terms of computer time when implementing our proposed procedure with respect to implementing the procedure proposed by Wall and Stoffer (2002).

Chapter 4

Summary of Conclusions and Future Research

The uncertainty associated with parameter estimation is present in almost all empirical work. In particular, it is an important component associated with the uncertainty of forecasts of future values of a variable of interest. In this thesis, we focus on unobserved component models and consider bootstrap methods to incorporate the parameter uncertainty on the prediction intervals of future values of the variables of interest and of the unobserved components of the model. Furthermore, bootstrap procedures not only incorporate the parameter uncertainty but they are also attractive because, in general, they do not rely on particular assumptions of the error distribution. Through this thesis, we use simulated and real time series data to illustrate the main results. Next, we describe the main contribution of the thesis.

In Chapter 2 we propose a parametric and a non-parametric bootstrap procedures for incorporating the uncertainty associated with parameter estimation into the prediction mean squared errors of the estimates of the unobserved components. By distinguishing between unconditional and conditional prediction mean squared errors, the proposed procedures are computationally easier than alternative bootstrap procedures previously available in the literature. Furthermore, we also carry out Monte Carlo experiments to compare the new procedures proposed in this thesis with the standard PMSE generated by the Kalman filter, with methods based in the asymptotic distribution of the parameter estimator and with alternative bootstrap procedures. We show that in the cases considered in the simulations (a random walk plus noise model with homoscedastic Gaussian, heteroscedastic and non-Gaussian errors), our procedures have better small sample properties. In addition, we show in an empirical application that when the uncertainty associated with parameter estimation is not taken into account, the conclusions about economic policy can change significantly.

In Chapter 3 we propose a new bootstrap procedure for incorporating the parameter estimation uncertainty into the prediction densities and, consequently, into the prediction intervals for future values of the series of interest. The proposed procedure is simpler than alternative bootstrap procedures because it avoids using the backward representation and incorporates the parameter and noise uncertainties in a single step. In addition, by carrying out Monte Carlo experiments, we analyze the finite sample performance of the proposed bootstrap prediction intervals and compared them with the standard intervals and alternative bootstrap intervals previously proposed in the literature. These experiments show that our bootstrap procedure has better small sample size performance when it is implemented in a random walk plus noise model. Moreover, it seems to have good properties for non-Gaussian disturbances, in particular for a χ^2_1 disturbance. The main results of Chapter 3 have been published in Rodriguez and Ruiz (2009).

Different topics, that have arisen while working on this thesis, are part of the future research agenda. We list the most relevant ones below.

- When dealing with the PMSE of the estimates of the unobserved states, we have seen that the bootstrap procedures proposed in this thesis have adequate properties even in non-Gaussian models. However, decomposition given in equation (1.5) is approximated by assuming that the cross-product $\underset{t-1}{E}\left[\left(\widehat{a}_{t|t-1} - a_{t|t-1}\right)\left(a_{t|t-1} - \alpha_t\right)'\right]$ is zero, and this result is only valid under Normality. Therefore, we want to analyze whether it is possible to obtain a better bootstrap approximation to the PMSE of the unobserved components in non-Gaussian models by incorporating an additional term taking into account the above expectation. Furthermore, although we propose PMSE which are robust in the presence of non-Gaussian errors, the intervals for the unobserved components are finally constructed by using the Normal quantile. Therefore, a natural extension is to extend the bootstrap procedures to obtain prediction intervals of the unobserved components that do not rely in any way on the Gaussianity of the errors.

- In Chapter 3, when we propose a bootstrap procedure to incorporate the parameter uncertainty into the prediction densities of future values of the series of interest, we assume that the system matrices are time-invariant. It could be of interest to study how to deal with time-variant systems. In this case, we could extend our procedures to models in which the disturbances are, for example, conditionally heteroscedastic. Furthermore, we have considered only prediction of future values of univariate time series, consequently, a natural extension is to consider the multivariate models.

- We want to consider the application of the bootstrap procedures proposed in this thesis for constructing densities and prediction intervals for returns and volatilities in stochastic volatility models.

- Finally, it is important to analyze the theoretical behavior of the procedures proposed in this thesis. Deriving their asymptotic properties is also in our further research agenda.

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